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Supergeometry, Supergravity and Kaluza-Klein Theory

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Abstract

We discuss methods of consistent Kaluza-Klein truncations of theories of supergravitation. We pay close attention to the underlying gauge and super-gauge theories supporting gravity and supergravity as is required for a thorough analysis of the compactification mechanisms. We specifically consider the case of Freund-Rubin compactification of $D = 11$ supergravity down to the maximally supersymmetric product space $\mathrm{AdS}_4 \times S^7$, with local SO(8) symmetry. This is motivated by the serendipitous features processed by $D = 11$ supergravity and its compactification, as well as the utility of the maximally symmetric space S^7.

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Chapter 1

Introduction

Unifying the forces of nature is a primary ambition of fundamental theoretical physics. Some of the largest landmarks in our understanding of the universe have come through discoveries of unification: from Maxwell unifying electricity and magnetism and Einstein unifying space and time, to Salam, Glashow and Weinberg unifying the electromagnetic and weak forces in what lead to the ultimate unification of forces in today's standard model. Supergravity (SUGRA) is no exception to this ambition, especially as gravity is the unruly sibling of the other, quantum, theories of nature. Supergravity brings together the principles of supersymmetry (SUSY) and general relativity (GR). However, much can be taken away from the name 'supergravity', with particular emphasis on the 'gravity'; as we discuss in this thesis, many of the analyses of supergravity take origin or heavy inspiration from corresponding principals developed for gravity.

There are many reasons to study supergravity. Not only does it attempt to bring gravity into the quantum fold, but it presents many physically interesting properties. For example, it sets an upper limit $D = 11$ to the physically viable theories of supergravity [1], and in this maximal case specifies a completely unique Lagrangian [2]. Moreover this dimension happens to be the minimal required to house the gauge groups of the standard model [3]. There is yet more, certain methods of reducing this theory to a physical theory of 4-dimensions pick out 4-dimensional subspace as the only choice [4]. Away from the physically inviting aspects of supergravity, it is common knowledge that general relativity recruits the help of geometry in its description of the universe, supergravity is beginning to return the favour. Supermanifolds as the geometric support to theories of supersymmetry and, in particular, local supersymmetry have become increasingly more researched. With the underlying algebraic concepts being ever refined [5, 6, 7], there are many more avenues for an increased understanding of such theories coming across the bridge between mathematics and physics.

The remit of this thesis lies in the pre-superstring era. As fascinating, rich and indeed modern as this topic is we have chosen a more narrow focus on the development of supergravity up to the point (or limit) that it meets superstring theory. The topic of primary interest to this paper is that of Kaluza-Klein (KK) reduction. This method can be considered the flagship of lowering dimensions of higher dimensional theories - a necessity if one is to take higher-dimensional theories seriously. Moreover, KK theory gives greater meaning to the idea that spacetime is geometry. It ultimately posits that all symmetries in nature may be cast as spacetime symmetries using the tools of gauge theory. The same is true when one considers supergravity as a supermanifold, where supersymmetry is now a super-spacetime supersymmetry. The methods of KK theory when applied to supergravity can be given additional clarity and intuition when dealing with supermanifold theory, as we attempt to show. We have thus taken a geometry and supergeometry-centric approach to KK theory, both to make the comparisons and analogy between gravity and supergravity clearer and to develop greater literacy of both theories.

The structure of this thesis is as follows. We begin by introducing the prinicples of gauge theory and Cartan geometry in chapter 2. We discuss the concept of fibre bundles, and how they are the correct description of local (gauge) symmetries. We then cast gravity as a Cartan geometry - a space locally modelled on a group *coset space* - to help develop the understanding and motivation required to introduce the supergravity Lagrangian in chapter 3. Following the concepts developed here we introduce the notion of a supermanifold and how they provide a clearer understanding of supersymmetry and, specifically, *local* supersymmetry. We introduce super-gauge theory and, in a similar vein to the preceding chapter, use it to introduce super-Cartan geometry and supergravity. We finish this chapter by writing the fundamental Lagrangian for supergravity. From here, in chapter 4, we up the dimensionality of supergravity. We analyse more abstractly what supergravity theories look like and we will discover that they have the property of a maximal and unique form in $D = 11$, for which we write down its Lagrangian. In the final chapter we look at methods of reducing the dimensionality of the theories considered before. We reveal Kaluza-Klein theory by finding general solutions to higher-dimensional space that admit a physically relevant factoring into submanifolds. We then look at the conditions under which the extraneous dimensions are irrelevant. We do this first by looking at the prototypical example of KK theory for gravity in $D = 5$. From here, we discuss the process more generally including how to deduce the form of the manifold factorisation as well as how to apply KK theory to supergravity theories. We finish the thesis by considering the maximally symmetric example of KK reduction on the $\mathrm{AdS}_4 \times S^7$ background and show how it successfully produces maximal supersymmetry in $D = 4$.

Chapter 2

Gravity as Cartan Geometry

The purpose of this section is to motivate the formulation of supergravity as super-Cartan geometry. Supergravity is the supersymmetric extension of gravity, and gravity can be formulated as Cartan geometry, so, to appreciate super-Cartan geometry, one ought to have an appreciation for 'civilian' Cartan geometry.

2.1 Fibre bundles

General relativity is a theory of Local Lorentz (LL) symmetry; gauge theory, to a physicist, is also a theory with local symmetries. It is, therefore, not a huge leap for one to conceive of a link between gravity and gauge theory. In fact, physical gauge theory and general relativity (in the first-order formalism with no torsion constraint) can be considered as specific examples of the *mathematical* area of study, also known as gauge theory, which is the study of geometric spaces known as fibre bundles [8]. As such, one can recast general relativity with the key descriptors of gauge theory, which we now introduce. To begin, the fundamental concept of gauge theory is the fibre bundle.

Definition 2.1 (Fibre bundle)**.** A fibre bundle is a structure $(P, \mathcal{M}, \pi, F)$, where P is a topological space known as the *bundle space*, $\mathcal{M}$ is also a topological space known as the *base space*, π is a continuous map $\pi : P \to \mathcal{M}$ called the *projection*, and a space F called the *fibre* of the bundle [9].

Intuitively, one can think of a fibre bundle, at least locally, as a given space, $\mathcal{M}$, where at each point, x defined on $\mathcal{M}$ there is a space isomorphic to F. Then one can picture the space with lines extending unfaithfully from x to each point in the fibre. Locally this is described as a *product space*:

$$\pi^{-1}(U) \simeq U \times F \mid U \subseteq \mathcal{M}, \tag{2.1}$$

where the set of all such maps defined on all open sets in the atlas of $\mathcal{M}$ is called the *local trivialisation* of P. The fibre itself can be any space, but we restrict our attention to those most relevant to gravity, namely the *vector* bundle (in pure gravity this can be further refined to the *tangent* bundle, but this is not true when one considers spinors in curved space, for example), the *principal* bundle and a cross-pollination of the two, an *associated* bundle. These now receive formal, but intuitive, definitions.

Definition 2.2 (Vector bundle)**.** A fibre bundle $(P, \mathcal{M}, \pi, V)$, where P is called a *real vector bundle* if V is isomorphic to a real vector space. That is to say that the local trivialisation is given by the following homomorphism [9, 10]:

$$\varphi : U \times \mathbb{R}^k \to \pi^{-1}(U) \mid U \subseteq \mathcal{M} \tag{2.2}$$

such that

$$v \mapsto \varphi(\{x\}, v) \mid \forall(\{x\}, v) \in (U,\, \mathbb{R}^k) \tag{2.3}$$

is linear, one-to-one and onto (i.e. bijective). The set notation should be taken to mean that for fixed $x \in U$ the resulting space is a vector space.

Comments.

1. An important case of a vector bundle is when the fibre is isomorphic to the tangent space of $\mathcal{M}$. This is called the *tangent bundle.*

2. Vector bundles are the tool one uses to equip manifolds with vector fields which transform in different representations of, say, a Lie group. One can think of equipping the manifold with an appropriate representation space at each point over the manifold. More on this later. The corresponding action of the Lie group then prompts the definition of a principal bundle.

Definition 2.3 (Principal bundle)**.** For a topological group G, a fibre bundle $(P, \mathcal{M}, \pi, G)$ is called a *G-principal bundle* if there is a continuous right action $P \times G \to P$ that is free (every point is changed under action of $g \in G$ unless $g = e$) and the coset space $P/G \simeq \mathcal{M}$. The group G is then called the *structure group* of the bundle [9, 11].

Comments.

1. The requirement of the existence of a right action is so that the fibre preserves its properties as a group, otherwise there is no distinction from a generic fibre bundle.

2. The above definition means that the local trivialisation of a principal bundle yields the structure $\mathcal{M} \times G \simeq P/G \times G$, which implies the action of G is transitive over the fibres (any point within a fibre can be reached from any other point within that same fibre by action of G). This means that each point $x \in \mathcal{M}$ is a stabiliser of G and so the entire group action is contained at each x. One can therefore interpret a principal bundle in the same way as a vector bundle, but where the fibre is now isomorphic to the structure group.

We now have two bundle structures which involve vector spaces and groups, respectively. It would then seem intuitive to unite the two to produce representations bundles. The role of such spaces is played by associated bundles and are, for example, how one constructs spinor fields in curved space.

Definition 2.4 (Associated bundle)**.** Take a principal G-bundle, $(P, \mathcal{M}, \pi, G)$, and a vector space, V, with a left G-action defined on it: $G \times V \to V$. We define the *associated bundle* of P to be the bundle $P \times V$ with equivalence relation defined, point-wise, by $(p, v) \simeq (pg, g^{-1}v)$. This is denoted $P_V \equiv P \times_G V$. Defining a projection map $\pi_V([p, v]) \equiv \pi(p)$ gives the overall bundle structure as $(P_V, \mathcal{M}, \pi_V, V)$.

Comments.

1. This definition implies that there is a homeomorphism $\pi_V^{-1}(x) \simeq V$ for all $x \in \mathcal{M}$, i.e P_V is a vector bundle with fibre V, equipped with a left G-action that is defined to act on the fibres. A proof of this can be found in A.1.

2. Putting this in the language of a physicist, an associated bundle allows one to define a representation of a group G which acts on the vector space V at each point on the space $\mathcal{M}$. Choosing $\mathcal{M}$ to be a space-time manifold, $G = \mathrm{Spin}^+(1, 3)/\mathbb{Z}_2$, and V to be Minkowski space, thus defines spin-$\frac{1}{2}$ fields in curved space.

2.2 Connections on fibre bundles

Before we discuss Cartan geometry and gravity, there is one more tool of great use to physicists that must be introduced - the connection. Loosely speaking, the connection encodes how fibres change 'orientation' with respect to one another as one moves over the manifold; it encodes curvature.

Definition 2.5 (Connection)**.** Given a principal G-bundle, $(P, \mathcal{M}, \pi, G)$, a *connection* is a smooth decomposition of the tangent space $T_pP = H_pP \oplus V_pP$ over every point $p \in P$, where H_pP is called the *horizontal* subspace, and V_pP the *vertical* subspace. The decomposition obeys the following conditions:

$$(a) \quad T_pP \simeq H_pP \oplus V_pP \quad \forall p \in P \tag{2.4}$$

$$(b) \quad \delta_{g*}(H_pP) = H_{gp}P \quad \forall g \in G,\ \forall p \in P, \tag{2.5}$$

where δ_g is the right action of g on P [9].

Comments.

1. The first condition is the statement that the decomposition holds at every point over the manifold meaning every tangent vector over P has this decomposition available. The second condition is the statement that the decomposition is preserved under the group action on the bundle (horizontal vectors are mapped to horizontal vectors etc.).

2. It has already been shown that the right action of g is free and transitive on the fibres of a principal bundle, and so the decomposition can be viewed as reflecting the local trivialisation of the bundle, i.e, for $P \simeq U \times G$ for all $x \in U \subseteq \mathcal{M}$, $T_pP \simeq T_x\mathcal{M} \oplus T_gG$ [12]. However, the tangent space of a Lie group is everywhere isomorphic to the tangent bundle at identity, or the Lie algebra [6]. So the decomposition can be written as $T_pP \simeq V_pP \oplus H_pP$ where $V_pP \simeq Lie(G) \equiv \mathfrak{g}$ and $H_pP \simeq T_{\pi(p)}\mathcal{M}$.

3. The connection can be canonically realised with a $\mathfrak{g}$-valued one-form, $\omega^a_\mu dx^\mu$ over P. This can be viewed as a machine that takes in $\tau \in T_pP$ and outputs $A \in \mathfrak{g}$ for $\tau \in V_pP$ and 0 otherwise, i.e., for $\tau \in H_pP$ (see definition 2.6 below). This is simple to understand; ω is $\mathfrak{g}$-valued from the start and so an identification with V_pP can be made with the inner product, where one discards $\tau \in H_pP$. Formally this is encoded in the following conditions:

$$(a) \quad \omega_p(V^A) = A, \ \forall(V^A, \ A, \ p) \in (V_pP, \ \mathfrak{g}, \ P) \tag{2.6}$$

$$(b) \quad (\delta^*_g\omega)_p(\tau) = Ad_{g^{-1}}(\omega_p(\tau)), \ \forall(p, \ g, \ \tau) \in (P, \ \mathfrak{g}, \ T_pP) \tag{2.7}$$

$$(c) \quad \tau \in H_pP \text{ iff } \omega_p(\tau) = 0. \tag{2.8}$$

4. Conditions 2.6 and 2.8 are self-explanatory. Condition 2.7 is a formulation of condition 2.5 using the transformation rule for the Lie algebra under the action of g, reaffirming the quality that only the vertical subspace transforms under group action.

This description of a connection is still not quite of the kind used to describe geodesics in general relativity. This is because we are currently only working with the connection defined over the *bundle*. What we really need is the *local representative*, the pullback, of ω over neighbourhoods, U, on the base space, $\mathcal{M}$. The pullback of ω is unusual in that it is defined relative to values in $Lie(G)$. Moreover, exactly what value in the Lie algebra the connection should take at each point is somewhat of a free parameter and one must make a specific choice to successfully define a pullback of the connection. This freedom of choice is what is known in gauge theory as a *section*. The example of tangent vector fields in the context of a tangent bundle makes the notion of a section more clear; the tangent bundle contains within it every possibility of tangent vector that could be defined over the base manifold. Selecting a specific tangent vector from each fibre at each point on the manifold defines a vector field over the manifold and is then a section of the tangent bundle - a slice of the possible tangent vector fields offered from the tangent bundle. Returning to the present example of connections, making a choice of section from a principal G-bundle allows one to define a connection with values in $Lie(G)$ determined by the embedding of $\mathcal{M}$ in G, given by the section, σ

$$\sigma : U \to P \simeq U \times G \tag{2.9}$$

$$\sigma^*\omega(T_{\sigma(x)}P) \simeq \sigma^*\omega(T_gG) = \omega^U(T_x\mathcal{M}), \tag{2.10}$$

where ω^U is the pullback of ω over the neighbourhood $U \subseteq \mathcal{M}$ [9]. 2.10 displays the freedom of choice in the pullback; the values $\tau \in T_{\sigma(x)}P$ are deduced by first pushing forward vectors in U by σ which then obviously depend on the choice of section. Hence the pullback correspondingly depends on ω. The notion of a section is important for making a gauge transformation of the connection. Immediately prior to the details of a gauge transformation we introduce the *Maurer-Cartan form*, which is a representation of the map mentioned in comment 3, above.

Definition 2.6. The *Maurer-Cartan form*, Ξ is the $Lie(G)$-valued one form over G which maps vectors in the tangent space $v \in T_gG$ to elements of the tangent space at identity (the Lie algebra), $v_e \in T_eG \simeq Lie(G)$, given by the left action of G, $l_g : h \mapsto gh \in G \mid \forall g, h \in G$ [9]

$$\langle \Xi, v \rangle(g') = l_{g'*}(l_{g^{-1}*}v). \tag{2.11}$$

Comments.

1. The Maurer-Cartan form can be understood by thinking of a Lie group as an ordinary manifold with tangent spaces defined everywhere over the space. Due to the relevance of the Lie algebra in generating the connected part of a group, other tangent spaces rarely enter the discussion. However, given that T_eG is isomorphic to the set of left-invariant vector fields, there is clearly a latent relationship between T_eG and tangent spaces at other points over G, T_gG. This relationship is built into the Maurer-Cartan form.

2. In words, 2.11 gives the value of a left-invariant vector field at the point g' which has v as its element at the point g.

Theorem 2.1. Let σ_1 and σ_2 be two local sections defined over a coordinate patch U:

$$\sigma_1 : x \mapsto (x, g_1) \in U \times G \tag{2.12}$$
$$\sigma_2 : x \mapsto (x, g_2) \in U \times G. \tag{2.13}$$

Let the two sections be related via

$$\sigma_2(x) = \sigma_1(x)\Omega(x), \;\; \Omega(x) \in G. \tag{2.14}$$

Then the local representatives of the connection ω defined using σ_1 and σ_2 are related by the gauge transformation

$$A_{(2)\mu}(x) = Ad_{\Omega(x)^{-1}}(A_{(1)\mu}(x)) + (\Omega^*\Xi)_\mu(x), \tag{2.15}$$

where Ξ is the Maurer-Cartan form of G.

Proof. See [9]. □

Comments.

1. Using 2.12 and 2.13 as examples, one can view sections as describing certain intrinsic information about the base space. In a spatial setting, as in general relativity, this can be related to the relative orientation of tangent spaces over a spacetime manifold. When the orientation is restricted to those given the Levi-Civitae connection one has the second order formulation of gravity. Keeping the orientation as an arbitrary element of the structure group gives the first order formulation.

2. The transformation law 2.15 can be understood as describing the change of the $Lie(G)$ values of the connection and the change of the isomorphism represented by the connection, between $v \in T_x\mathcal{M}$ and $v_g \in T_gG$ as prescribed by the choice of section. The Maurer-Cartan form then enters to convert a covector into the change of Lie algebra incurred by the change in section. This is better illustrated in the following sequence

$$T_x\mathcal{M} \xrightarrow{\Omega_*} T_{\Omega(x)}G \xrightarrow{\Xi} Lie(G) \tag{2.16}$$

2.3 Parallel transport and covariant differentiation

With the definition of a connection in hand we can introduce the notion of parallel transport and the covariant derivative in gauge theory. The local representatives of the covariant derivative play a pivotal role in defining the action in theories of gravity, supergravity and all physical gauge theories in general.

As mentioned in definition 2.5, there is an isomorphism $\pi_* : H_pP \to T_{\pi(p)}\mathcal{M}$. Then, given a vector field X over $\mathcal{M}$ one may define a curve in P, denoted $X^\uparrow$, that may intuitively be considered identical to X.

Definition 2.7 (Horizontal lift - Vector field)**.** Given a principal bundle, $(P, \mathcal{M}, \pi, G)$, and a vector field X over $\mathcal{M}$, there exists a corresponding unique vector field $X^\uparrow$ in P called the *horizontal lift* defined such that [9]

$$(a) \quad \pi_*(X_p^\uparrow) = X_p \in \mathcal{M}, \; \forall p \in P \tag{2.17}$$
$$(b) \quad \mathrm{ver}(X_p^\uparrow) = 0, \; \forall p \in P. \tag{2.18}$$

Comments.

1. There is a freedom to how one embeds X_p into T_pP stemming from the fibre of the bundle, i.e. one could drop the condition 2.18 such that ver$(X_p^\uparrow)\in V_pP$. This could, however, be considered a redundancy of the embedding - the gauge freedom. The horizontal lift can therefore be interpreted as introducing no additional parameter with the embedding.

Given the definition of the horizontal lift of a vector field it is natural to restrict the field to vectors defined along a curve. This then defines the horizontal lift of the curve itself in P with tangent vectors equal to the horizontal lift of the tangent vectors defined over the curve in $\mathcal{M}$.

Definition 2.8 (Horizontal lift - Curve)**.** Given a principal bundle, $(P, \mathcal{M}, \pi, G)$, let C be a curve on $\mathcal{M}$, $C : \mathbb{R} \supset [a, b] \to \mathcal{M}$. Then the *horizontal lift* of C, $C^\uparrow : [a, b] \to P$, is a curve in P such that $\pi(C^\uparrow(t)) = C(t)$, $\forall t \in [a, b]$ and the tangent vectors to $C^\uparrow$ are all horizontal, in the sense of definition 2.7 [9].

The horizontal lift is of central importance when defining both parallel transport, and covariant differentiation. The horizontal lifts of a curve allows one to ask the question of how fibres change parameterised *strictly* by the base manifold. In a sense the change incurred of a fibre over the base manifold due to a gauge transformation is removed so that vectors in different fibres can be honestly compared. The general idea behind horizontal lifts and parallel transport is captured by the following theorem [9]

Theorem 2.2. Given a principal bundle, $(P, \mathcal{M}, \pi, G)$, and curve, $C(t)$, for each point $p \in \pi^{-1}(C(t)) \subset P$ there is a *unique* horizontal lift of C such that $C^\uparrow(t) = p$.

Proof. Let σ be a section of P and C be a curve over $\mathcal{M}$. Define the non-horizontal lift of C as the image of σ over C, $\sigma(C(t)) = \gamma_\sigma(t) \in \pi^{-1}(C(t))$. Define the horizontal lift associated with σ by $\tilde{\gamma}(t) = \gamma_\sigma(t)g(t)$, where $g(t) \in G$. Given $\pi^{-1}(a) \simeq G$, $\forall a \in \mathcal{M}$, for each section, σ, the values of $g(0)$ are in one-to-one correspondence with the value of $\sigma(C(0))$, each specifying an initial condition for $g(0)$. Now pushforward 2.15 using $\tilde{\gamma}(t)$, which can be recast as a map, $\tilde{\gamma}(t) = \sigma(C(t))g(t) : \mathcal{M} \to P$. Applying a vector tangent, which by definition is horizontal, and using the $\omega(X_p) = 0$, $\forall X_p \in H_pP$ gives

$$0 = \mathrm{Ad}_{g(t)^{-1}}(\omega_{\gamma_\sigma}([\gamma_\sigma])) + \Xi_{g(t)}([g]), \tag{2.19}$$

where $[\gamma_\sigma]$ denotes the equivalence class of curves through, i.e tangent vectors at, points over γ_σ. Finally, if G is a matrix group 2.19 can be written as

$$0 = g(t)^{-1}\omega_{\gamma_\sigma}([\gamma_\sigma])g(t) + g(t)^{-1}\frac{dg}{dt}(t). \tag{2.20}$$

This is first order a differential equation for g, which by the existence and uniqueness of Ordinary Differential Equations (ODEs) means that, for a given initial condition $g(0)$ which specifies a condition for $\tilde{\gamma}(0)$ the horizontal lift defined by 2.20 is unique [9, 12]. $\square$

Having defined a set of unique horizontal lifts of a given curve C over $\mathcal{M}$ one can make clear the notion of parallel transport in a principal bundle.

Definition 2.9 (Parallel transport)**.** Let $C : \mathbb{R} \supset [a, b] \to \mathcal{M}$ be a curve over $\mathcal{M}$. Let $C^\uparrow$ be the unique horizontal lift of C which passes through the point $u_a \in \pi^{-1}(C(a))$ a la theorem 2.2. The *parallel transport* of u_a along C is the map

$$\Gamma : \pi^{-1}(C(a)) \to \pi^{-1}(C(b)),\ u_a \mapsto u_b, \tag{2.21}$$

where u_b lies along the horizontal lift of C which passes through u_a [9, 12].

Comments.

1. This is a map between end points of a horizontal curve in P, meaning both points have horizontal tangent vectors, hence they are *parallel* with respect to the vertical subspace of the tangent bundle, TP.

This definition and theorem 2.2 allows one to define curves in P equivalent to a curve, C, in $\mathcal{M}$ based on different embeddings of C in P, by essentially separating the gauge redundancy from the curve itself. The next step associated to this would be to assess what 2.20 implies for sections of the associated fibre bundle, i.e., to swap the sections defined by $g(t)$ with the corresponding vectors in the bundle upon which the structure group G acts. This is the machinery required for covariant differentiation.

Definition 2.10 (Horizontal and vertical subspaces of associated bundles)**.**

1. Let ω be a connection over the principal G-bundle $(P, \mathcal{M}, \pi, G)$, and let $(P_F, \mathcal{M}, \pi_F, F)$ be the associated fibre bundle of G. Then ω defines a vertical and horizontal decomposition of the tangent bundle, TP. Define the vertical subspace of $T_y P_F$, $y \in P_F$ by
$$V_y P_F \equiv \{\tau \in T_y P_F \mid \pi_{F*}\tau = 0\}. \tag{2.22}$$

2. Let $B_v : P \to P_F$, $v \in P_F$ be defined as $B_v(p) \equiv [p, v], p \in P$ using the equivalence class of definition 2.4. Define the horizontal subspace of $T_{[p,v]}P_F$, $[p, v] \in P_F$ by
$$H_{[p,v]}P_F \equiv B_{v*}(H_p P), \tag{2.23}$$
where $H_p P$ is the horizontal subspace defined by ω [9].

Comments.

1. Developing 2.23 more, this condition implies that the local trivialisation of the tangent space $T_{(x,g)}P \simeq T_x U \oplus T_g G, \forall x \in U \subset \mathcal{M}$ is carried over to $T_{[p,v]}P_F$ such that $H_{B_v((x,g))}P_F \simeq T_x U$.

2. With the notion of a horizontal subspace defined on the associated fibre bundle, one can define the notion of both a horizontal lift and parallel transport on P_F as follows. Let $C^\uparrow$ be the unique horizontal lift of the curve $C : \mathbb{R} \supset [a, b] \to \mathcal{M}$ to P, passing through $C^\uparrow(a) = p$. Then let $[p, v] = [C^\uparrow(a), v]$ be the corresponding point in P_F. By 2.23 the curve
$$C_F^\uparrow(t) \equiv B_v(C^\uparrow(t)) = [C^\uparrow(t), v] \tag{2.24}$$
is the horizontal lift of C to P_F passing through $[C^\uparrow(a), v] = [p, v]$. This leads to a notion of parallel transport, as note that under the projection $\tau : [p, v] \mapsto v$ is constant over $C_F^\uparrow(t)$ [9].

3. Given a choice of horizontal lift, i.e for a specific boundary condition $C_F^\uparrow(a) = [C^\uparrow(a), v]$ one can define the image of $C_F^\uparrow(t)$ in the local trivialisation as $\chi : P_F \to U \times F$ by
$$\chi : C_F^\uparrow(t) \mapsto (C(t), g(t)v), \tag{2.25}$$
where $g(t)$ obeys 2.20. This is a statement of the possible embeddings of C into P_F in correspondence to a choice of section for a given initial condition [9].

This last statement all but defines covariant differentiation, by giving a view to the possible sections that a curve over $\mathcal{M}$ can admit i.e, the possible gauge freedoms acting on vectors defined over the curve. Using this, one can compare vectors laying in different fibres which are related by a gauge transformation, g. Equation 2.20 relates how the gauge transformations vary over the curve, which can then be related to how associated fibres vary over the curve. The associated fibre section act as a proxy for the gauge group section over the curve in $\mathcal{M}$. Thus, by parallel-transporting a section of a fibre over a curve, one can remove the interference caused by the gauge group and honestly compare elements in different fibres, by essentially making all fibres 'equal'. This makes a derivative of the fibres an element of a fibre and hence transforms under the left action of the structure group, hence the use of *covariant*. This is represented by the following formal definition of a covariant derivative.

Definition 2.11 (Covariant derivative)**.** Let $(P, \pi, \mathcal{M}, G)$ be a principal G-bundle and V be a vector space transforming as a representation of G. Take $C : [0, \epsilon] \to \mathcal{M}$ with $\epsilon > 0$ to be a curve in $\mathcal{M}$ with boundary condition $C(0) = x_0 \in \mathcal{M}$, and let $\psi : \mathcal{M} \to V$ be a section of the associated vector bundle, $(P_V, \pi_V, \mathcal{M}, V)$. The *covariant derivative* of ψ along the curve C at the point x_0 is defined as
$$D_C\psi \equiv \lim_{t\to\infty} \left(\frac{\Gamma_V^t \psi(C(t)) - \psi(x_0)}{t} \right) \in \pi_V^{-1}(x_0), \tag{2.26}$$
where Γ_V^t is the inverse parallel transport map, $\Gamma_V^t : \pi_V^{-1}(C(t)) \to \pi_V^{-1}(x_0)$.

Comments.

1. Recall that one associates a horizontal lift with a given initial condition of the section ψ. Therefore the map Γ^t_V associates points to boundary conditions uniquely.

2. This definition is defined as an element of the vector bundle. To finally produce the form one is familiar with, we map it to a local representative over $\mathcal{M}$. Let ψ be a section associated with $\sigma : \mathcal{M} \to P$ over P given by $\psi(x) = [\sigma(x), \psi_U(\sigma(x))]$, where $\psi_U : [p, v] \mapsto v \in V$ is the *local representative* of $\psi(x)$; $\psi_U(x)$ is a vector in V locally associated with element $g \in G$ of the section $\sigma(x) = (x, g)$ $\forall x \in U \subset \mathcal{M}$. So,

$$\begin{aligned} \psi(t) &= [\sigma(t), \psi_U(\sigma(t))] = [C^{\uparrow}(t)g(t)^{-1}, \psi_U(C(t))] \\ &\simeq [C^{\uparrow}(t), g(t)^{-1}\psi_U(C(t))], \end{aligned} \tag{2.27}$$

where the isomorphism is that used in definition 2.4. This yields

$$\Gamma^t_V \psi(C(t)) = [C^{\uparrow}(0), g(t)^{-1}\psi_U(C(t))] \tag{2.28}$$

as the parallel transport of the $\psi(C(t))$ by the section $\sigma(t)$, with local representative of $g(t)^{-1}\psi_U(C(t))$. Thus, the covariant derivative of $\psi_U(C(t))$ at $t = 0$ with boundary condition $g(0) = e$ is given by

$$\frac{d}{dt}\left(g(t)^{-1}\psi_U(C(t))\right)\Big|_{t=0} = \left(\frac{d}{dt}\left(g(t)^{-1}\right)\psi_U(x_0) + g(t)^{-1}\frac{d}{dt}\left(\psi_U(C(t))\right)\right)\Bigg|_{t=0}. \tag{2.29}$$

With a some simple algebraic manipulation using 2.20 this can be massaged into the following familiar form [9]

$$D_\mu\psi(x) \equiv \partial_\mu\psi(x) + A_\mu(x)\psi(x). \tag{2.30}$$

Just as the derivative of a function can be viewed as a particular case of the more general definition of exterior derivatives of forms (the derivative of a function is the same as the exterior derivative of a 0-form), the covariant derivative (2.30) can be viewed as a particular case of a more general definition, unsurprisingly called the *covariant exterior derivative*. The starting point of this definition is by considering a vector field, $\psi(x)$, to instead be a vector-valued 0-form, i.e,

$$\psi^a(x) = \psi^a \otimes f(x) \in V \otimes \Omega^0(P) \equiv \Omega^0(P, V) \tag{2.31}$$

The generalisation is clear; this definition can be extended to a vector-valued k-form via an abuse of notation

$$\phi^a_{\mu_1\ldots\mu_k}(x) = \phi^a \otimes z_{\mu_1\ldots\mu_k}(x) \in V \otimes \Omega^k(P) \equiv \Omega^k(P, V). \tag{2.32}$$

In analogy with the philosophy behind the covariant derivative, simply taking the exterior derivative of a vector-valued form is no longer tensorial, due to the vectorial component responding to a change in tangent space of the bundle. If we are to make the exterior derivative covariant one must compensate for the change of ϕ^a incurred by moving from point to point on the bundle. This can be achieved in two equivalent ways, explicit details of which can be found in [9, 12].

1. For a vector-valued one form, we restrict the full exterior derivative over the principal bundle to only take horizontal vectors as arguments. Thus, the domain of the form when viewed as a map lies in H_pP which is isomorphic to $T_{\pi(p)}\mathcal{M}$. Overall this defines an isomorphism between H_pP and V, thus encoding data about how ϕ^a varies strictly over the manifold, guided by $H_pP \simeq T_{\pi(p)}\mathcal{M}$. This first method can be explicitly accomplished by starting with the full exterior derivative of $\phi^a_{\mu_1\ldots\mu_k}(x)$ and then formally subtracting any contributions from vertical vectors. For a connection 1-form, ω, and for vector space V being a representation ρ of G the result is [13, 14]

$$\mathrm{d}_D\phi \equiv \mathrm{d}\phi + \rho(\omega) \wedge \phi, \tag{2.33}$$

where the wedge product applies to arguments only, i.e, $(\rho(\omega) \wedge \phi)(X, Y) = \rho(\omega(X)) \cdot \phi(Y) - \rho(\omega(Y)) \cdot \phi(X)$ and e.g. $\rho(\omega(X))$ acts on the vector $\phi(Y)$.

2. Replace the regular derivative in the definition of the exterior derivative with the covariant derivative. This makes all derivatives elements of V, thus forcing it to be covariant.

$$
\begin{aligned}
(\mathrm{d}_D\phi)(X_0,...,X_{k+1}) = & \sum_{i=0}^{k+1}(-1)^i D_{X_i}\phi(X_0,...,\hat{X}_i,...,X_{k+1}) \\
& + \sum_{i<j}(-1)^{i+j}\phi([X_i,X_j],X_0,...,\hat{X}_i,...,\hat{X}_j,...,X_{k+1}),
\end{aligned} \tag{2.34}
$$

where $\hat{X}_i$ means X_i has been omitted as an argument [14].

These two interpretations can be tied into the same formal definition as follows

Definition 2.12 (Covariant exterior derivative)**.** Let ϕ be a vector-valued k-form $\phi \in V \otimes \Omega^k(P) \equiv \Omega^k(P,V)$. The *covariant exterior derivative* is the map defined by

$$
\begin{gathered}
\mathrm{d}_D : \Omega^k(P,V) \to \Omega^{k+1}(P,V), \\
(\mathrm{d}_D\phi)(X_1,...,X_{k+1}) \equiv (\mathrm{d}\phi)(\mathbf{hor}(X_1),...,\mathbf{hor}(X_{k+1})),\ \forall X_i \in T_uP,
\end{gathered} \tag{2.35}
$$

where $\mathbf{hor}(X_i)$ denotes the projection of tangent vector X_i onto the horizontal subspace of T_uP.

The exterior covariant derivative is how curvature is defined in gauge theory. Formally, we define the curvature associated with a connection 1-form, ω, to be its exterior covariant derivative. One may interpret the exterior contribution as encoding the differences between how vectors are changed when moving over different curves on a manifold, while the covariance restricts this to be a property of the manifold rather than its embedding in the principal bundle, thus defining the *curvature of the manifold* $\mathcal{M}$.

Definition 2.13 (Curvature 2-form)**.** Let $(P,\pi,\mathcal{M},G)$ be a principal bundle equipped with connection 1-form, ω. The curvature 2-form associated with ω is given by its covariant exterior derivative

$$
\mathrm{d}_D\omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega], \tag{2.36}
$$

where $[\omega \wedge \omega] = \omega^a \wedge \omega^b [T_a, T_b]$ implies adjoint action between the *Lie*(G) component and the wedge product between the 1-form component.

Comments.

1. Making a choice of local section to define the connection, $\sigma : U \to P$ allows 2.36 to be pulled back to the base manifold $\mathcal{M}$ giving

$$
\mathrm{d}_D A^c = \sigma^*(\mathrm{d}_D\omega) = \mathrm{d}A^c + \frac{1}{2}A^a \wedge A^b C^c_{ab} \tag{2.37}
$$

$$
= (\frac{1}{2}\partial_{[\mu}A^c_{\nu]} + \frac{1}{2}C^c_{ab}A^a_\mu A^b_\nu)e^\mu \wedge e^\nu \tag{2.38}
$$

$$
\equiv \frac{1}{2}F^c_{\mu\nu}e^\mu \wedge e^\nu, \tag{2.39}
$$

where in the final line, $F^c_{\mu\nu}$ has been defined as the local representative of the curvature of ω - or the field strength. This can, alternatively, be written as

$$
F^c_{\mu\nu} = \partial_\mu A^c_\nu + C^c_{ab}A^a_\mu A^b_\nu = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + [A_\mu, A_\nu]^c \tag{2.40}
$$

With the tools of gauge theory defined, we now consider the case of structure groups G which have an invariant subgroup, H, the quotient by which leads to a principal H-bundle, G/H, known as a *Klein geometry.* We consider how such quotient spaces can be used to build arbitrary principal H-bundles which infinitesimally resemble quotient spaces G/H. The study of such spaces is called *Cartan geometry.*

2.4 Klein and Cartan geometry

In short, Cartan and Klein geometries are the study of principal G-bundles which are isomorphic, or locally isomorphic, to coset manifolds given by the quotient of G by H, G/H. A coset manifold is a many-to-one representation of a Lie group and a manifold upon which the group G 'naturally' acts, manifesting as the Killing vector fields of that space. Coset manifolds and Killing vectors will be of great importance when we finally discuss Kaluza-Klein theory in section 5.1. Put in reverse order, manifolds can be characterised by their isometry groups. This is the sense which pertains to Klein geometry. Considered as a space, each $g \in G/H$ represents a point and the subgroup G/H acts transitively, while H is the stabiliser group of each point. In other words, neglecting the action of G/H itself, the manifold has the structure of a principal H-bundle [15].

Definition 2.14 (Klein geometry)**.** Let G be a Lie group with Lie subgroup H such that the quotient G/H is a connected space. A *Klein geometry* is defined as the pair (G, H) [6].

An important case to consider, being one that applies to Minkowski space, de Sitter and anti-de Sitter space written as Klein geomtries as $(\mathrm{ISO}(\mathbb{R}^{1,3}), \mathrm{SO}^+(1,3))$, $(\mathrm{SO}(1,4), \mathrm{SO}^+(1,3))$ and $(\mathrm{SO}(2,3), \mathrm{SO}^+(1,3))$, is when the Lie algebra G is reducible as an H-module, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$. In this case the Klein geometry is referred to as *metric* and *reductive*. Metric reductive Klein geometries are useful when one wishes to generalise to curved spaces, or Cartan geometries, which we introduce in a matter of moments. In any case the subalgebra $\mathfrak{g}/\mathfrak{h}$ is invariant under the group action of H, the adjoint action, $\mathrm{Ad}(H)$ and the Klein geometry hence has the structure of a principal H-bundle with associated tangent vector bundle $\mathfrak{g}/\mathfrak{h}$. With $\mathfrak{g}/\mathfrak{h}$ being the tangent space of the coset manifold, the action $\mathrm{Ad}(H)$ defines a *frame bundle* on G/H with structure group H. This is also known as an *H-structure* on the manifold [16, 17]. Hence, with such a structure flat spacetime can be defined in terms of Klein geometry. It is then natural to imagine how this formalism can be extended to curved spacetimes, which we now understand as infinitesimally modelled on Klein geometry. The study of such spaces is *Cartan geometry.*

Definition 2.15 (Cartan geometry)**.** Given a Klein geometry (G, H), a *Cartan geometry* is an H-principal bundle $(P, \pi, \mathcal{M}, H)$ equipped with a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ referred to as the *Cartan connection* that satisfies the following properties [6]

$$(a) \quad A_p(X_p) = X \; \forall X \in \mathfrak{h} = T_e H, p \in P \tag{2.41}$$

$$(b) \quad \Phi_h^* A = \mathrm{Ad}_{h^{-1}}(A) \; \forall h \in H \tag{2.42}$$

(c) It obeys the *Cartan condition*, $A_p : T_p P \to \mathfrak{g}$ is an isomorphism $\forall p \in P$. (2.43)

Comments.

1. There are important distinctions to be made between the Cartan connection and a principal connection, introduced in definition 2.5. While A is $\mathfrak{g}$-valued and could be endowed with the structure of a principal G-connection, as our bundle is an H-principal bundle it is not necessarily a principal H-connection. Rather, it simply offers an isomorphism $T_x\mathcal{M} \simeq \mathfrak{g}/\mathfrak{h}$. Therefore, starting with a G-principal bundle, and restricting the structure group to the subgroup H, the Cartan connection maps the moduli space $\mathfrak{g}/\mathfrak{h}$ to $T_x\mathcal{M}$. This thus ensures that $\mathcal{M}$ is everywhere a Klein geometry, with tangent space correctly identified as $\mathfrak{g}/\mathfrak{h}$.

2. When the underlying Klein geometry is metric and reductive then the Cartan connection obeys a corresponding split

$$A = p_{\mathfrak{g}/\mathfrak{h}}(A) + p_{\mathfrak{h}}(A) \equiv e + \omega, \tag{2.44}$$

where p_i is the projection onto the i^{th} subspace. It follows in this case that the $\mathfrak{h}$-valuedness of the connection can be retained even while considering only the moduli space $\mathfrak{g}/\mathfrak{h}$. Thus the space is now equipped with a principal H-connection as well as the isomorphism outlined in the preceding remark, i.e, a *vielbein field*, hence the notation 'e' as commonly seen in the tetrad formalism of GR.

3. For the limiting case of flat Cartan geometry or rather Klein geometry we see that the would-be Cartan connection is simply the Maurer-Cartan connection of before. This is obvious when considering the restatement that the Cartan connection is a map of tangent spaces to *Lie*(G) and the fact that for Klein geoemtry, the *whole* space can be seen as $\in G$ rather than just locally.

We have now reached a intersection between the gauge theory notion of a connection and the Cartan connection. We can thus begin to link the familiar tools of gauge theory to Cartan geometry, and with it build up the dynamics of a theory with local diffeomorphism invariance, i.e, general relativity. With the Cartan connection, we can consider the associated *Cartan curvature*

$$F(A) \equiv \mathrm{d}A + \frac{1}{2}[A \wedge A]. \tag{2.45}$$

Note that, although is schematically the same as the exterior covariant derivative of a principal connection, this is not the same curvature as in the sense of equation 2.36. This is because A is not a principal connection; it is not equivariant under the action of $g \in G$ despite having values in $\mathfrak{g}$. This notion curvature contains information about how the Cartan geometry deviates from the Klein geometry it is based on. To see this, consider the Maurer-Cartan form, Ξ of the underlying Klein geometry. It can be shown [18] that the Cartan curvature is identically zero

$$F(\Xi) = \mathrm{d}\Xi + \frac{1}{2}[\Xi \wedge \Xi] = 0. \tag{2.46}$$

Therefore the extent to which equation 2.45 is non-zero, characterises the deviation of the Cartan geometry from the 'flat' Klein geometry. This is key to the notion of Cartan geometry being a deformation of Klein geometry. Klein geometry is characterised strictly by elements of a group, structurally represented by equation 2.46. For Cartan geometry, this is not globally true, the degree to which is represented by equation 2.45.

Next we can consider what happens to the Cartan curvature when we have a metric reductive Cartan geometry. This yields the decomposition [6]

$$F(A) = p_{\mathfrak{g}/\mathfrak{h}}(F) + p_{\mathfrak{h}}(F) \equiv F(\omega) + E^{(\omega)} + \frac{1}{2}[e \wedge e]. \tag{2.47}$$

Given the decomposition of A into a vielbein field and a genuine principal H-connection, equation 2.47 contains the curvature 2-form of ω,

$$F(\omega) = d_D\omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega], \tag{2.48}$$

as well as terms involving the vielbein field. In particular, the cross-term

$$E^{(\omega)} = \mathrm{d}e + [\omega \wedge e] = \frac{1}{2}T_{bc}{}^{a}\theta^{b} \wedge \theta^{c} \tag{2.49}$$

is the *torsion 2-form* of the connection ω [12, 6].

So, it can be seen that using Cartan geometry, for the privileged spaces modelled on Klein geometries, one can endow the H-reduction of a principal bundle with a both a vielbein field and principal connection in one move. Conversely, given a space with a vielbein field and principal connection which is locally a Klein geometry, we can contextualise this with in the framework of Cartan geometry. This is the important point when considering general relativity, which involves spaces modelled on the Klein geometry of Minkowski spacetime $(\mathrm{ISO}(\mathbb{R}^{1,3}), \mathrm{SO}^{+}(1,3))$. We can now make the statement that such a Cartan geometry defines an $\mathrm{SO}^{+}(1,3)$ reduction of the bundle of possible frames of the tangent spaces of a manifold $\mathcal{M}$. The space also comes with the structure of an H-principal connection with associated curvature, a vielbein field and the possibility of torsion. The question is now: what should be done with these components to build a physical theory? Since we are insisting on the view of GR as a gauge theory, we would like to keep as many as the raw ingredients of Cartan geometry as possible, with the constraint that they yield the 'regular' form of gravity (i.e. the Einstein-Hilbert (EH) action, or second order formulation). Under this condition, the ansatz for gravity within the back-drop of Cartan geometry, for a section $\sigma : \mathcal{M} \subseteq U \to P$, is

$$S[A] = \int_{\mathcal{M}} \sigma^{*}(F(\omega)^{ab} \wedge e^{c} \wedge e^{d})\epsilon_{abcd}, \tag{2.50}$$

where ϵ_{abcd} is the Levi-Civita symbol. The section, σ is necessary define a theory over the base manifold, $\mathcal{M}$ as we are still working with a principal bundle. It is important to note that currently the e and ω are independent tensor fields. The action is therefore minimised with respect to both these fields. The resultant field equations are both respectively first order equations, hence the nomenclature 'first order formulation gravity'.

To summarise the content of this chapter, the local lorentz invariance at the core of general relativity can be modelled on the geometry of fibre bundles. In particular, general relativity can be modelled on Cartan geometry owing to the fact that the spacetimes of interest in GR are locally isomorphic to coset spaces of an associated Lie group, G. Using the machine of Cartan geometry, one can then produce all the relevant geometric quantities used in GR (principal connection and vielbein), but in a more general manner where they are independent. Gravity can then be reformulated in terms of these newfangled quantities. The first-order formalism is of great importance when one considers fermionic fields in curved space, which are naturally described using vielbeins, something necessary to devise supergravity theories. As such we now consider the super-analogue of the Einstein-Cartan perspective of gravity and introduce super-Cartan geometry for supergravity over supermanifolds.

Chapter 3

Supergravity as Super-Cartan Geometry

The key to motivating the form of supergravity theories originates from gravity being a gauge symmetry. Qualitatively speaking, if one is to fit the metric tensor into a supersymmetric multiplet, then the fact that supersymmetry blends bosonic and fermionic fields into one another necessarily causes any local symmetry of the bosonic field to bleed into the fermionic fields. Thus, supergravity is intrinsically a *local* supersymmetry. Conversely, if one wishes to supersymmetricaly extend the local Lorentz symmetry, by closure of the Lie superalgebra, the symmetry is naturally local [1]. Starting with Cartan geometry as ones understanding of gravity, promoting the Lie algebra of the Lorentz symmetry to a corresponding super-Lie algebra leads to the analogous concept of super-Cartan geometry. Broadly, this is the notion that, theories of supergravity are locally isomorphic to 'flat' super-coset spaces. This purpose of this chapter will be to formalise super-Cartan geometry and thus motivate supergravity. This will be useful when discussing the truncation of higher-dimensional supergravity in later sections.

3.1 Local Lorentz transformations

Before we can consider supergauge theory, we must motivate the fact that the supersymmetric extension of gravity, *necessarily* has local supersymmetry. We begin by looking explicitly at the form of LL transformations: these are a kind of *diffeomorphism*, or reparameterisation, of the underlying manifold. Structurally, they have the familiar form of a Taylor expansion over the corresponding coordinate patch. Working with an infinitesimal coordinate transformation, f, a generic scalar field transforms as:

$$\begin{gathered} f : x^\mu \mapsto x^\mu - \epsilon^\mu(x) \\ \phi(x) \mapsto \phi(x) + \epsilon^\mu(x)\partial_\mu\phi + O(\epsilon^2) \end{gathered} \tag{3.1}$$

where the dependence of ϵ on x denotes that this is a local transformation. This is important for the structure of supergravity. The exact form of 3.1 will depend on the form of the coordinate transformation given in the first line. For this reason it is more instructive to rewrite 3.1 in terms of symmetry group generators, T_α:

$$\phi(x) + i\theta^\alpha(x)T_\alpha\phi + O(\theta^2). \tag{3.2}$$

Such a transformation law only constitutes a classical field transformation. In quantum theory, however, fields are promoted to operators and so an appropriate transformation law must be found that binds the transformation (3.2) to how quantum operators must transform. This leads to the identification:

$$i\theta^\alpha[\hat{T}_\alpha, \phi] = i\theta^\alpha\hat{T}_\alpha\phi = \delta\phi(x). \tag{3.3}$$

The left-hand side of (3.3) is what will be taken to mean an infinitesimal change, henceforth.

3.2 Local supersymmetry

For the heuristic purpose of introducing super-Cartan geometry, while the dimensions we work with are kept arbitrary, we restrict our consideration to $\mathcal{N}=1$ supersymmetry. Maximal supersymmetry and its corresponding consequences on dimensionality will be considered after the fact. Thus, we start with the super-Poincaré algebra (anti-)commutation relations for what we initially assume is $\mathcal{N}=1$ *global* supersymmetry:

$$\begin{aligned} \{\mathcal{Q}_\alpha, \dot{\mathcal{Q}}^\beta\} &= -\frac{1}{2}(\gamma_\mu)^\beta_\alpha P^\mu \\ [M_{\mu\nu}, \mathcal{Q}_\alpha] &= -\frac{1}{2}(\gamma_{\mu\nu})^\beta_\alpha \mathcal{Q}_\beta \\ [P^\mu, \mathcal{Q}_\alpha] &= 0 \end{aligned} \tag{3.4}$$

Performing a *local* Lorentz transformation δ_L, followed by a *global* supersymmetry transformation, δ_S, on a scalar bosonic operator gives the infinitesimal change, and vice versa:

$$\delta_S\delta_L\phi(x) = i\eta^\alpha[\mathcal{Q}_\alpha, \phi(x)] - \eta^\alpha\omega^{\mu\nu}(x)[\mathcal{Q}_\alpha, [M_{\mu\nu}, \phi(x)]] \tag{3.5}$$
$$\delta_L\delta_S\phi(x) = i\omega^{\mu\nu}(x)[M_{\mu\nu}, \phi(x)] - \eta^\alpha\omega^{\mu\nu}(x)[M_{\mu\nu}, [\mathcal{Q}_\alpha, \phi(x)]]. \tag{3.6}$$

Developing 3.5 further:

$$\begin{aligned} \delta_S\delta_L\phi(x) &= i\eta^\alpha[\mathcal{Q}_\alpha, \phi(x)] + \eta^\alpha\omega^{\mu\nu}(x)([M_{\mu\nu}, [\phi(x), \mathcal{Q}_\alpha] + [\phi(x), [\mathcal{Q}_\alpha, M_{\mu\nu}]] && (3.7)\\ &= i\eta^\alpha[\mathcal{Q}_\alpha, \phi(x)] - \eta^\alpha\omega^{\mu\nu}(x)[M_{\mu\nu}, [\mathcal{Q}_\alpha, \phi(x)]] + \frac{1}{2}\eta^\alpha\omega^{\mu\nu}(x)(\gamma_{\mu\nu})^\beta_\alpha[\phi(x), \mathcal{Q}_\beta] && (3.8)\\ &= i(\eta^\alpha + \frac{i}{2}\eta^\beta\omega^{\mu\nu}(x)(\gamma_{\mu\nu})^\alpha_\beta)[\mathcal{Q}_\alpha, \phi(x)] - \eta^\alpha\omega^{\mu\nu}(x)[M_{\mu\nu}, [\mathcal{Q}_\alpha, \phi(x)]] && (3.9)\\ &= i\bar{\eta}^\alpha(x)[\mathcal{Q}_\alpha, \phi(x)] - \eta^\alpha\omega^{\mu\nu}(x)[M_{\mu\nu}, [\mathcal{Q}_\alpha, \phi(x)]], && (3.10) \end{aligned}$$

where the Jacobi identity has been used to go from 3.7 to 3.8, and the SUSY algebra and anti-symmetry of commutation relations have been used to reach 3.10. The final term in 3.10 can be recognised as the same as the final term in 3.6 allowing the following substitution to be made:

$$\delta_S\delta_L\phi(x) = i\bar{\eta}^\alpha(x)[\mathcal{Q}_\alpha, \phi(x)] + \delta_L\delta_S\phi(x) - i\omega^{\mu\nu}(x)[M_{\mu\nu}, \phi(x)] \tag{3.11}$$
$$[\delta_S, \delta_L]\phi(x) = i\bar{\eta}^\alpha(x)[\mathcal{Q}_\alpha, \phi(x)] - i\omega^{\mu\nu}(x)[M_{\mu\nu}, \phi(x)]. \tag{3.12}$$

Then 3.12 can be recognised as the infinitesimal change induced by the transformation:

$$e^{i(\bar{\eta}^\alpha(x)\mathcal{Q}_\alpha - \omega^{\mu\nu}(x)M_{\mu\nu})}\phi(x)e^{-i(\bar{\eta}^\alpha(x)\mathcal{Q}_\alpha - \omega^{\mu\nu}(x)M_{\mu\nu})}. \tag{3.13}$$

Since this transformation was derived using initially global SUSY transformations, this transformation must also be a symmetry but with local parameterisation, $\bar{\eta}^\alpha(x)$. Thus, 3.13 can be recognised as a general, *local* SUSY transformation, with $\bar{\eta}^\alpha(x)\mathcal{Q}_\alpha - \omega^{\mu\nu}(x)M_{\mu\nu}$ recognised as a general element of the super-Poincaré algebra.

In addition, the local Grassman parameter, which has been defined by:

$$\bar{\eta}^\alpha(x) = \eta^\alpha + \frac{i}{2}\omega^{\mu\nu}(x)(\gamma_{\mu\nu})^\alpha_\beta\eta^\beta \approx (e^{\frac{i}{2}\omega^{\mu\nu}(x)\gamma_{\mu\nu}})^\alpha_\beta\eta^\beta, \tag{3.14}$$

is the Lorentz transformation rule for a spinor. Therefore, the locality of a Lorentz transformation has, coupled by the fact that spinors transform in representations of the Lorentz group, induced a local SUSY transformation, 3.14.

3.3 Supergeometry

With local supersymmetry established as intrinsic to any supersymmetric theory of gravity, we can build up the analogy between the ordinary gauge theory and Cartan geometry of chapter 2, and super-Cartan geometry. Prior to this, however, we must introduce the notion of a supermanifold as the generalisation of *superpace*. Superspace can be largely accredited to the work of Salam and Strathdee [19, 20]. The idea is to extend bosonic space by introducing spatial fermionic degrees of freedom one for each supercharge. This is in analogy with the relationship between spacetime degrees of freedom and momentum operators; the two imply each other, to translate in a spatial direction obviously requires the presence of a degree of freedom. Then, the fermionic or 'odd' degrees of freedom can be viewed as the fundamental space on which the supercharges act, and superspace as a whole facilitating supersymmetry. Using this viewpoint one can build up the physically most intuitive interpretation of a supermanifold, developed by Rogers [21], in response to [19, 20], and deWitt [22]. This interpretation is largely developed by analogy with the physical understanding that ordinary *bosonic* manifolds locally resemble flat space. The work of [21] built up a mathematically precise definition of a supermanifold based on the flat superspace of [19, 20]. There is an alternative understanding of supermanifolds, the so-called *algebro-geometric approach* developed by Berezin, Lietes and Kostant [23, 24], which is less accessible to the practical uses of a physicist. This method is the analogue of the perspective that manifolds can be viewed as locally-ringed spaces; supermanifolds are locally super-ringed spaces. The reason why this is less useful to physics, and therefore to the present understanding of supergravity, is that the notion of locally-ringed spaces requires no reference to specific points over the manifold. A supermanifold is defined only by the fact it is endowed with local ring structure, which require only the notion of open sets. Although the two formulations are related, we restrict our discussion of supermanifolds to that developed by Rogers [21] for the reasons outlined.

3.3.1 The deWitt-Rogers supermanifold

As mentioned, this notion of a supermanifold makes use of definite points. One extends ordinary e.g. Minkowski spacetime with additional degrees of freedom that, crucially, anti-commute. A priori there is no need for these coordinates to obey non-trivial anti-commutation relations, such as those of the super-Lie algebra. This is because these coordinates need only support the action of supercharges, which requires them to be, at minimum, elements of a Grassmann algebra. A super-Lie algebra structure then navigates movement within these coordinates, in complete analogy with the role of generators on Minkowski space. This concept will be explored formally when we discuss super-Cartan geometry. More abstractly, superspace can be defined from an underlying Grassman algebra. Following the notation of [21], we denote this algebra as $B = B_0 \oplus B_1$, where the 0^{th} component is *even* (commuting) and the 1^{st} component is *odd* (anti-commuting), as standard for a Grassmann algebra. By nature of this division, this defines a $\mathbb{Z}_2$ graded vector space. One then uses this algebra to build up the desired-dimension flat space as Cartesian products of the odd and even components, denoted by $B^{m|n} = B_0^m \times B_1^n$. Such a space is then parameterised by coordinates denoted $(x_1, ..., x_m|\xi_1, ..., \xi_n)$. From here, the motivations behind ordinary manifolds hold true: one wishes to model curved supermanifolds on flat superspace such that each section is woven smoothly together. Thus, we require a super-analogue of charts and transition functions, and thus a notion of differentiability on superspace.

Definition 3.1 (Superdifferentiation). Let f be the map $f : B_0^m \times B_1^n \supseteq U \to B$. Then f is called *superdifferentiable* if there exists maps $D_i f : B_0^m \times B_1^n \supseteq U \to B$ such that the limit

$$\lim_{y\to 0} = \frac{||f(x+y) - f(x) - \sum y^i D_i f(x)||}{||y||} = 0 \tag{3.15}$$

exists, where $x, y \in U$ [21].

To gain an intuition of what this definition implies, we build up a polynomial expansion of superdifferentiable and, further, *superanalytic superfunctions*, f. Consider the projection operations

$$P_i : \; B_0^m \times B_1^n \to B, \; (x_1, ..., x_{m+n}) \mapsto x_i, \tag{3.16}$$

where x_i briefly denotes all superspace coordinates. Recall that the even component of B is commuting. This is enough to establish a homomorphism referred to as the *body map*, b, between algebras $b : B \to \mathbb{R}$, thus there exists projection maps $b \circ P_i \equiv r_i : U \to \mathbb{R}$. We can extend this to the full projection of B onto real space

$$r : B_0^m \times B_1^n \to \mathbb{R}^m, \; (x_1, ..., x_m|\xi_1, ..., \xi_n) \mapsto (r_1(x_1), ..., r_m(x_m)). \tag{3.17}$$

Next, consider a basic smooth function, $f \in C^\infty(r(U)) \otimes B$ which also takes values in B, where C^∞ is the set of smooth (infinitely differentiable) functions over $r(U) \subseteq \mathbb{R}^m$. By analytically continuing the f into the domain B_0 one forms a generic expansion for the possible superfunctions over (currently) B_0

$$\hat{f}(x;\xi) \equiv \sum_{i_1=0,...,i_m=0}^{L} \frac{1}{i_1!...i_m!}[(\partial_1^{i_1}...\partial_m^{i_m})f(r(x;\xi))] \times s(x_1)^{i_1}...s(x_m)^{i_m}. \tag{3.18}$$

The map $s : x_i \mapsto x_i - b(x_i)$ is called the *soul map* and projects in this case to the even elements of B generated by anti-commuting generators of B. This function is clearly analytic as (a) $f \in C^\infty$ and (b) due to the finite nature of B, after L powers of $s(x_i)$ the terms vanish. We now need to account for the possible ways one can use the odd coordinates, ξ_i, to construct objects over the supermanifold. Due to the direct product structure of B, for each function of the form 3.18, we can multiply by an element of B_1. Generically this means that we take B_1-valued functions 3.18 and we can write a general expansion of a superfunction as

$$f(x;\xi) = \sum \hat{f}_\mu(x)\xi^\mu., \tag{3.19}$$

where the sum is over all possible combinations of the generators ξ_i. Any function which admits such an expansion is *supersmooth* or *superanalytic* and is labelled as an element $f \in G^\infty(U)$ [5]. Note that because we have taken B to be a finite Grassmann algebra any repeats of odd coordinates in the expansion 3.19 vanish due to their anti-commuting properties. This causes ambiguities in this expansion as the components $\hat{f}_\mu(x)$ are determined depending on the initial choice of Grassmann algebra, B. The ambiguities can be solved by either taking an infinite-dimensional algebra such that the sum will never 'artificially' truncate, or reinterpreting the ambiguities as reparamterisations of the same physical theory; changing algebra should not change the physical theory similar to a gauge redundancy.

We now have a definition of a basic supersmooth functions as maps $B_0^m \times B_1^n \supseteq U \to B$, that is, from superspace to a superpoint. Using these constituent functions one can build up supersmooth transition functions $B_0^m \times B_1^n \supseteq U \to U' \subseteq B_0^{m'} \times B_1^{n'}$. Thus, one may define a supermanifold by equipping a topological space, X, with charts and transition functions of this kind in a globally consistent manner, in analogy with an ordinary manifold. Such a supermanifold with G^∞ transition functions is referred to as a G^∞*-supermanifold*, or a *deWitt-Rogers supermanifold*.

With the super-analogue of a manifold defined we are in a position to fully extend the analogy of Klein and Cartan geometry based on Lie groups to their super-counterpart based on super-Lie groups. We shall see that supermanifolds will facilitate local supersymmetry transformations and how the abstract behaviour of supermanifolds can be brought back down to the familiar ground of spacetime manifolds by way of fermionic and bosonic fields.

3.4 Super-Klein and super-Cartan geometry

Given that a supersymmetry algebra is a $\mathbb{Z}_2$-graded vector space, if one considers a space on which the super-Lie algebra acts one needs both commuting and anti-commuting parameters, i.e. that the generators must be an element of the space upon which they act. Thus, interpreting this space as a super-Lie group, in parallel to how one may interpret the parameters of a Lie group, we see that it is necessarily a supermanifold. Therefore one concludes, like the definition of a Lie group, that a super-Lie group is a supergroup that is also a supermanifold. Formally [5],

Definition 3.2 (Super-Lie group)**.** Let G be a type G^∞-supermanifold equipped with group structure defined by G^∞ operations of type

$$\begin{aligned} G \times G \to G, &\quad (g_1, g_2) \mapsto g_1 g_2 \\ G \to G, &\quad g \mapsto g^{-1} \end{aligned} \tag{3.20}$$

where the latter operation ensures a unique inverse, if bijective. Then, G is a *super-Lie group.*

Given flat superspace upon which G, but more generally any supermanifold, is based one may define derivatives of odd coordinates by using the algebra of derivations. That is, by refining the information of a derivative to just its algebraic properties and applying it to a superalgebra. This is key when one wishes to define a vector over curved superspace which can be viewed as derivations of functions evaluated at a point. Therefore, the definition of a vector on a supermanifold is analogous to the definition over a regular manifold, with the caveats that (a) the derivations have a graded structure, reflective of the graded structure of the underlying supermanifold and (b) that they will also be a B-module (which strictly makes them vector-valued B-modules), where B is the underlying Grassmann algebra upon which supermanifold is based. Thus with a vector structure successfully applied to a supermanifold, we can extend this to a vector field over the space just as one does in differential geometry. Returning to super-Lie groups, it is next natural to consider the set of left-invariant vector fields over the supermanifold. The left-invariant quality of these vector fields provides an isomorphism to (by convention) the tangent space at identity, referred to as the *super-Lie algebra*, $\mathcal{L}(G)$. That super-Lie algebras possess a closed bilinear map

$$\begin{aligned} [\cdot\,,\cdot]:\ &\mathcal{L}(G)\times\mathcal{L}(G)\to\mathcal{L}(G) \\ &(X,Y)\mapsto[X,Y]=XY-(-1)^{|X||Y|}YX,\ \forall X,Y\in\mathcal{L}(G) \end{aligned} \tag{3.21}$$

follows from the same justifications that the Lie bracket emerges for regular group manifolds, with the gradation emerging due to the gradation of the super-Lie algebra. Thus, one can see that the relationship between super-Lie groups and super-Lie algebras is analogous to the one between Lie groups and Lie algebras, and so can be interpreted in much the same way. Once again, we have the caveat that super-Lie algebras are in general graded B-modules and so have the structure $B\otimes\mathfrak{g}$, a property inherited from their construction using Grassmann algebras, B. This begins to provide the motivation behind the use of supergeometry in describing the action of supergauge theories. On this note, we start our discussion of super-Klein and super-Cartan geometry.

The recipe for super-Klein and Cartan geometries follows in the footsetps of the regular theory, introduced in section 2.4. The results introduced here closely follow the work of Eder [6], formal proofs and greater detail on the subject can be found there.

Definition 3.3 (Super-Klein geometry)**.** Given a super-Lie group, G, with embedded super-Lie subgroup, H, such that there exists a connected subspace G/H, one defines the *super-Klein geometry* as the pair (G,H).

Comments.

1. The basic facts surrounding this definition follow in similar fashion to regular Klein geometry with the exception of occasionally appending the prefix 'super' to some words, but it is worth reemphasising them both as a reminder of the principals and also to see the specifics of the super-geometric case. The coset space G/H has the structure of a super-principal H-bundle with inherited right action of H, $G\times H\to H$.

2. One can define the super-analogue of a Maurer-Cartan form over the G-bundle by defining a $\mathcal{L}(G)$-valued 1-form. Taking X_i as a basis for the $\mathfrak{g}$-module $\mathcal{L}(G)\simeq\mathfrak{g}\otimes B$ and ω^i a corresponding dual basis of 1-forms, we define the *super-Maurer Cartan form*

$$\Xi_S\equiv\omega^i\otimes X_i. \tag{3.22}$$

   As a map between tangent spaces, the interpretation is the same as for the original Maurer-Cartan form, (2.11).

We now introduce a concept which is essential if one wishes to define *both* bosonic and fermionic fields as fields over a spacetime manifold, i.e. physical space. Grassmann-valued objects cannot be measured. They are merely a mathematical device used to codify physical phenomena, what one measures are real numbers. Thus all physical quantities extracted from a supergeometric theory must grant a description solely on a bosonic manifold. Whilst this presents no problem for bosonic quantities, the only way to describe fermionic quantities over a bosonic manifold is by encasing them inside a bosonic quantity. In other words, pairing them up with another Grassmann-odd object. Now, if one starts with a supermanifold, $\mathcal{M}$, then there is a bosonic submanifold, $\mathcal{M}_0\subseteq\mathcal{M}$. Thus, one can pullback quantities described over the full supermanifold, to the bosonic submanifold. The constraint physics imposes is that both Grassmann odd and even-valued objects can be pulled back to $\mathcal{M}_0$ with no dependence on the Grassmann odd sector of the supermanifold, i.e. that physics is completely determined by the bosonic submanifold. This principle is referred to as the *rheonomy principle* [25]. To abide this principal one should not consider a lone supermanifold, $\mathcal{M}$, rather a product of supermanifolds, $\mathcal{S}\times\mathcal{M}$, which together with a projection $pr_{\mathcal{S}}:\mathcal{S}\times\mathcal{M}\to\mathcal{M}$ is called an *s-relative* supermanifold denoted $\mathcal{M}_{/\mathcal{S}}$, for short [6, 26]. When taking a superfunction of an s-relative supermanifold, certain cross terms between respective even and odd contributions from each supermanifold yield bosonic packages of ferminoic components. We have already considered the closely related G^∞-superfunctions, however here one explicitly evaluates the B-valuedness and extracts only the resulting even elements and in general $\mathcal{S}$ can be any supermanifold although for practical purposes one usually considers the linear superspace B. The superfunctions one considers over $\mathcal{S} \times \mathcal{M}$ yield the superfields containing supermultiplets in any supersymmetric theory.

There are few intricacies that need to be considered when introducing this additional supermanifold, however. One wishes that the physical theory depends as little as possible on it since as its role is strictly to introduce a mix of bosonic *and* fermionic components when we restrict to the spacetime submanifold. Thus, as we now begin to consider Cartan geometry and principal superbundles in more detail we wish that this structure applies as much as possible to the 'primary' supermanifold $\mathcal{M}$. In general, the rubric for dealing with s-relative supermanifolds is to restrict all the relevant objects of gauge theory to the primary supermanifold, and from those definitions build up a supergauge theory of the s-relative supermanifold which is at all points isomorphic to a supergauge theory over just the primary manifold. The entire recipe for such constructions is laid out in [6, 26], and the interested reader is referred to there. For our purposes however, we extract only the key results. Thus, from super-Klein geometry, one can define a super-Cartan geometry using an s-relative principal bundle, modeled on a super-Klein geometry.

Definition 3.4 (Super-Cartan geometry)**.** Let $(P_{/\mathcal{S}}, \pi_{\mathcal{S}}, \mathcal{M}_{/\mathcal{S}}, H)$ be an s-relative principal superbundle with structure group H and (G, H) be a super-Klein geometry. Then a *super-Cartan geometry* modelled on (G, H) is the principal bundle $(P_{/\mathcal{S}}, \pi_{\mathcal{S}}, \mathcal{M}_{/\mathcal{S}}, H)$ equipped with an even $L(G)$-valued s-relative 1-form $\mathcal{A} \in \Omega^1(P_{/\mathcal{S}}, \mathfrak{g})_0$, that satisfies the following requirements

1. $\mathcal{A}(\mathbb{1} \otimes X^Z) = Z \ \ \forall Z \in \mathfrak{h}$
2. $\delta_h^* \mathcal{A} = \mathrm{Ad}_{h^{-1}} \mathcal{A} \ \ \forall h \in H$
3. Given an embedding map ϵ_P of a primary supermanifold, P, (choice of parameterisation) $\epsilon_P : P \hookrightarrow P \times S$, the pullback of $\mathcal{A}$ via this map defines an isomorphism $\epsilon_P^* \mathcal{A}_p : T_p P \to L(G)$.

Comments.

1. An even 1-form is chosen ultimately we are concerned with physics over a bosonic (even) submanifold.
2. The first two conditions are reminiscent of the conditions used to define a regular Cartan connection, definition 2.15. A difference here is that the isomorphism is only between vertical vectors over the primary supermanifold, $\mathbb{1} \otimes X^Z$, as one demands that the physics has as little dependence on the parameterising supermanifold as possible.
3. The final condition is the *super-Cartan condition*. The motivation for this is as for a regular Cartan connection, except here one also has the additional freedom of S. Once one has defined a particular embedding, the pullback offers the same interpretation as before: that the tangent space is everywhere $T_p P \simeq \mathfrak{g}/\mathfrak{h}$, as one understands what a super-Cartan geometry to be.

Once again, we are interested in a particular subset of super-Cartan geoemtries, those with super-Lie algebras that can be split into H-invariant subspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$. Such spaces are also referred to as *metric* and *reductive*. As in the ordinary case, there is a corresponding split in the super-Cartan connection which yields a genuine principal H-superconnection, ω, and a *supervielbein*, E

$$\mathcal{A} = p_{\mathfrak{g}/\mathfrak{h}}(\mathcal{A}) + p_{\mathfrak{h}}(\mathcal{A}) \equiv E + \omega. \tag{3.23}$$

In analogy with the role of a vielbein as a soldering form, the supervielbein bridges the gap between the tangent space of the supermanifold and the moduli space of the super-Klein geometry, thus 'soldering' the group action to the local supergeometry. Therefore, for metric reductive super-Cartan geometries there is the structure of local supersymmetry, which can now be readily applied to theories of supergravity.

Taking super-Minkowski space, $(\mathrm{ISO}(\mathbb{R}^{1,3|4}), \mathrm{Spin}^+(1,3))$, as our case study of a super-Klein geometry, using the universal cover $\mathrm{Spin}^+(1,3)$ to include a a real Majorana representation of the fermionic degrees of freedom, the super-Lie algebra splits as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h} = \mathfrak{spin}^+(1,3) \oplus \mathfrak{m} = \mathfrak{spin}^+(1,3) \oplus \mathbb{R}^{1,3} \oplus \Delta_{\mathbb{R}}. \tag{3.24}$$

Throughout, "1,3|4" refers to the direct sum "fundamental|real Majorana" representations of the Lorentz group, as indicated by the second equality in (3.24); $\Delta_{\mathbb{R}}$ denotes the real Majorana representation. Given the direct sum structure of the moduli space, $\mathfrak{m}$ the supervielbien admits a decomposition

$$E = Y + \psi = e^\mu P_\mu + \psi^a Q_a, \tag{3.25}$$

where we have written this as an element of the super-Lie algebra expanded in the basis of generators. As the connection is defined to be even, due to the rheonomy principle, one may deduce that $\psi \in \Omega^1_{hor}(P_{/S}, \Delta_{\mathbb{R}})_0$ and $e \in \Omega^1_{hor}(P_{/S}, \mathbb{R}^{1,3})_0$, where the 1-forms are horizontal with respect to the adjoint action of H [6]. Overall, and split in terms of generators and components, the super-Cartan connection 1-form reads

$$\mathcal{A} = \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu} + e^\mu P_\mu + \psi^a Q_a. \tag{3.26}$$

The results of regular Cartan geometry follow through: the supervielbein and its H-invariance, defines a H-reduction ($\mathrm{Spin}^+(1,3)$ reduction, here) of the frame bundle of the supermanifold. Restricting to the bosonic submanifold, this induces a spin-structure on the body manifold - a fermionic field defined over the manifold, which in this case also helps to furnish a representation of local supersymmetry, as will be seen shortly.

With the notion of an s-relative super-Cartan connection in hand we progress to introducing *super-Cartan curvature* and, with it, supergravity. The definitions of curvature and Cartan-curvature, 2.13, are, morally, the same. This is to be expected because, with regards to the abstract definitions of curvature, very little has changed; 1-forms have been replaced by super 1-forms, Lie algebras have been replaced by super-Lie algebras. The main things that need to be taken into account are (a) the (anti-)commuting properties of the supervector elements and (b) one wishes the parameterising manifold to have as little involvement in the definitions as possible. Applying these conditions, yields a very analogous result - a mathematically rigorous cover of this is given in [26]. The super-Cartan curvature of a super-Cartan connection, $\mathcal{A}$, is hence defined as [6]

$$F(\mathcal{A}) = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}] = \mathrm{d}\mathcal{A} + \frac{1}{2}(-1)^{|T_A||T_B|}\mathcal{A}^A \wedge \mathcal{A}^B \otimes [T_A, T_B], \tag{3.27}$$

where the wedge product takes the same definition as in equation (2.36), as shown by the second equality. The minus factor in front comes from anti-commuting $\mathcal{A}^B$ past T_A (the grade of $\mathcal{A}^B$ is the same as T_B). Following the split (3.24), the super-Cartan curvature can be separated into corresponding components. Using the SUSY algebra, it can be shown that the components reduce to

$$F(\mathcal{A}) = F(\mathcal{A})^\mu P_\mu + \frac{1}{2}F(\mathcal{A})^{\mu\nu} M_{\mu\nu} + F(\mathcal{A})^a Q_a \tag{3.28}$$

$$F(\mathcal{A})^\mu = \mathrm{d}Y^\mu + \omega^\mu{}_\nu \wedge Y^\nu - \frac{1}{4}\bar{\psi} \wedge \gamma^\mu \psi \equiv \Theta^{(\omega)\mu} - \frac{1}{4}\bar{\psi} \wedge \gamma^\mu \psi \tag{3.29}$$

$$F(\mathcal{A})^{\mu\nu} = F(\omega)^{\mu\nu} \tag{3.30}$$

$$F(\mathcal{A})^a = \mathrm{d}(\mathcal{A})^a + (\rho_{*\Delta_{\mathbb{R}}}(\omega) \wedge \psi)^a = D^{(\omega)}\psi^a. \tag{3.31}$$

Here ω; $\Theta^{(\omega)}$, $F(\omega)$ and $D^{(\omega)}\psi$ are the spin connection; its torsion, curvature and covariant derivative of ψ. Using these components, just as at the end of section 2.4, one can make an ansatz for a supergravity action, based on the Cartan geometry $(\mathrm{ISO}(\mathbb{R}^{1,3|4}), \mathrm{Spin}^+(1,3))$ modelled on super Minkowski space. The ansatz is formulated by taking a section over the pullback of the principal super-bundle, σ, to the bosonic submanifold, and assuming the most general diffeomorphism-invariant integral involving $F(\mathcal{A})$ over that section. Then the form of the integral is refined by restricting the diffeomorphisms to the relevant symmetry groups of supergravity. This method is outlined in e.g. [27]. The result is an action involving the components (3.29-3.31) as follows

$$S(\mathcal{A}) = \frac{1}{2\kappa}\int_M \sigma^*((\frac{1}{2}F(\omega)^{\mu\nu} \wedge e^\alpha \wedge e^\beta)\epsilon_{\mu\nu\alpha\beta} + i\bar{\psi} \wedge \gamma_* \gamma_\mu D^{(\omega)}\psi \wedge e^\mu). \tag{3.32}$$

This is, as promised, the action for $D = 4$, $\mathcal{N} = 1$ action for supergravity which is also the fundamental part of any supergravity action. We note that if one assumes that a trivial parametersing super manifold $\mathcal{S}$, i.e a point $\{*\}$, then there is, by definition, no fermionic component to $\mathcal{S}$. Thus there is no possible combination of fermionic components between $\mathcal{S}$ and P that will yield a Grassmann even variable and thus, when pulled back to the bosonic submanifold, will not contribute to the action (3.32), which thus reduces to equation (2.50) of ordinary Einstein gravity. This is not only a reassurance that the super-Cartan view of supergravity is an extension of the Cartan geometry of gravity, but it also demonstrates the necessity of a parameterising supermanifold in supergauge theory to yield fermionic components over a bosonic manifold. Conversely, if one starts with a supergravity theory, then if one is to contextualise it over a supermanifold, this necessarily requires the presence of the supermanifold $\mathcal{S}$.

We have thus motivated supergravity as super-Cartan geometry. The next step is to consider more general supergravity theories, in particular in higher dimensions, and to witness the relationship between dimenisionality and the permitted 'size' of the SUSY group. This will give rise to the idea of maximum dimension of supergravity theories, which ties in as the low energy limit of superstring theory. Then, once we reach the heights of $D = 11$ supergravity we consider what methods must be implemented to return the theory to the ground level reality of $D = 4$.

3.5 First-order formulation of supergravity

We finish this chapter by briefly introducing a concept that will be useful later on. Before this, though, we recap the main points discussed so far. Gravity has the structure of a gauge theory. Supergravity is the supersymmetric extension of gravity. It is therefore an obvious consequence that the vielbein of general relativity, which maps between different tangent space bases, is extended to a *supervielbein*, an isomorphism between super-vector spaces, defined at a point on a supermanifold [28]. By analogy with the role of vielbeins to Cartan connections, this means that the promotion of a vielbein to supervielbein induces a similar change in the connection, or gauge field in gauge theory terminology [6]. In plainer speaking, the Cartan connection of general relativity is extended by way of a fermionic field, the *gravitino*. The addition of the gravitino field, which ensures local supersymmetry, gives rise to the supergravity multiplet representation of supersymmetry. The Einstein-Hilbert action is thus extended by the introduction of an additional interaction term [1]:

$$\mathcal{S}_{E\text{-}H} = \frac{1}{2\kappa^2}\int d^D x\, e\, e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega), \tag{3.33}$$

$$\mathcal{S}_{gravitino} = -\frac{1}{2\kappa^2}\int d^D x\, e\, \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho, \tag{3.34}$$

$$\mathcal{S}_{supergravity} = S_{E\text{-}H} + S_{gravitino}. \tag{3.35}$$

The form of 3.34 can be understood, if one understands the gauge theory formulation of gravity, by analogy with 3.33 as the *exterior-covariant* derivative (the exterior product extracts curvature, covariance ensures that it is tensorial) of the gravitino soldering form; the gravitino action is just the local representative of the curvature 2-form of the gravitino field. Strictly speaking, 3.33 and 3.34 could be combined into the curvature of a supermultiplet, which would represent the supergauge theory of supergravity. However, splitting the equations up like this is physically more sensible as it is indicative of having defined bosonic and fermionic fields *over* a bosonic submanifold, i.e. they are fields over spacetime. This is the structure of a *super-Cartan geometry*, analogous to the description of gravity using Cartan geometry, see [6, 26] for formal details of this. Indeed, in reference to super-Cartan geometry, we note that 3.33-3.35 is the first order formulation of supergravity, where the spin connection, ω, is independent of the vielbein, e. The first order and second order formulations of gravity and now supergravity refer to the order of the differential equations that constitute solutions to the equations of motion [1]. In the present case, resorting to the first order formulation, one avoids derivatives of the veilbein that exist in the second order formulation and that stem from defining ω in terms of the veilbein. This can simplify the process of proving local supersymmetry, however other complications arise so in practice one tends to use an intermediate formulation known as the "1.5 order formulation" [1]. The first order formulation has been stated in 3.33 - 3.35 as a link to the interpretation of supergravity as super-Cartan geometry.

Chapter 4

Raising Dimensions

Higher-dimensional extensions of gravity lead to the inclusion of other gauge fields when localising back to a $D = 4$ manifold. This is one of the motivations behind the Cartan formalism of gravity and will be studied in detail in section 5. In similar spirit, the super-Cartan geometric view of supergravity ushers a better understanding of generalising the theory to higher dimensions and how allows one to incorporate other gauge groups into the theory. The purpose of this section is to introduce higher dimensional theories of supergravity, in particular the, as we shall see, maximal dimension $D = 11$ supergravity. Then, following the overall spirit of this paper, we return to regular gravity, this time in higher dimensions, to motivate why one should care about higher-dimensional theories at all as well as developing the techniques for truncating the extra dimensions and thus restoring physical sense to the abstract nature of a higher dimensional theory.

4.1 General construction of supergravity for $D \geq 4$

As any supersymmetric theory requires spinors, we begin with their general construction in any dimension. Spinors are defined as elements of a vector space which furnishes a representation of a Clifford algebra. For the physicists purposes, the Clifford algebra is defined using the anti-commutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}, \tag{4.1}$$

where $\mu \in \{0, ..., D-1\}$, the γ^μ are the generators of the algebra and $\eta^{\mu\nu}$ is the Minkowski metric in D spacetime dimensions. The Clifford algebra is then the space spanned by the γ-matrices. The general-dimension algebra can be constructed using the form in $D = 4$, which may be defined using the Pauli matrices and $\mathbb{1}_2$. It is first easiest to consider the relation (4.1) for $\eta^{\mu\nu} \leftrightarrow \delta^{\mu\nu}$, where the Pauli matrix anti-commutation relations fit more obviously into the general construction

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_2 \tag{4.2}$$

$$\begin{aligned}
\gamma^1 &= \sigma_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes ... \\
\gamma^2 &= \sigma_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes ... \\
\gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1}_2 \otimes ... \\
\gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1}_2 \otimes ... \\
\gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes ... \\
&...
\end{aligned} \tag{4.3}$$

The idea behind this construction is that each element must mutually anti-commute with each other. Thus, each γ-matrix must be separated by at least one anti-commuting element (i.e., one Pauli matrix). Hence why the element, e.g., $\sigma_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes ...$ is omitted as this does not anti-commute (rather it commutes) with the element γ^3. Then, to finish the generalisation of (4.1) rather than (4.2), one chooses any element from (4.3) and multiplies it by i, which then squares to $-\mathbb{1}$ [1].

One can see that each identity factor in (4.3) is replaced after every two γ matrices, thus giving the relation that for $D = 2m$, for m even, m factors in the tensor product are required. Each Pauli matrix is a 2×2 matrix and thus the overall representation therefore has dimension 2^m. For

$D = 2m+1$ we take the $(2m+1)^{th}$ element and truncate it to only the first m factors. This is acceptable as, for the elements $i \leq 2m+1$, the factors in the truncated 'isles' are exclusively identity matrices, so the anti-commutation properties between the first $2m+1$ γ matrices are contained in the first m factors. Thus for D even and odd, one has a 2^m dimensional representation. The Clifford algebra is then generated by the identity and the span of the elements γ^μ. Due to the defining anti-commutation relation (4.1) any symmetric product of the γ matrices can be reduced and so the only unique products that can be formed are antisymmetric ones.

$$\gamma^{[\mu_1}\gamma^{\mu_2}...\gamma^{\mu_r]} \equiv \gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_r}. \tag{4.4}$$

The number of products is not infinite, however; there is a highest rank product denoted by γ_{D+1} in even dimensions and γ_* in general

$$\gamma_* \equiv (-1)^{m+1}\gamma_0\gamma_1...\gamma_{D-1}. \tag{4.5}$$

For $D = 2m$, γ_* can be added to the set of $2m$ generators to form the set of γ-matrices for $D = 2m+1$.

Elements of a vector space that furnish a representation of the γ-matrices in arbitrary dimensions are then spinors. Furthermore, when one is constructing a theory with symmetry, it is typical to consider extremes: either minimal symmetry or maximal symmetries, the choice of an other kind of symmetry is an ambiguity. For this purpose, when building the most simple supersymmetry theories one uses the most simple kind of spinor: the Majorana spinor. In a 2^m-dimensional space, a Majorana has 2^{m-1} independent components. This is because Majorana spinors are spinors which are subject to a reality constraint; spinors in general are necessarily $\mathbb{C}$-valued and enforcing a reality condition halves the numbers of components. Note that this condition can only be met in certain spacetime dimensions. Crucially, however, Majorana spinors exist for $D = 11$. As we now see, this is important for determining the maximum dimension for allowed supergravity. Firstly, we note that the maximum allowed helicity, h, of a supersymmetry muliplet is $h = 2$. Higher helicity representations involve particles which cannot be coupled to a Lorentz-invariant current in a consistent way. Thus an interacting theory of such particles cannot be made, and as such would be unobservable and is not useful to consider. Now consider $D = 11$, we begin with the *simplest possible* supersymmetric theory, $\mathcal{N} = 1$, which involves the simplest possible spinor of a supergravity theory - a gravitino of type Majorana. Majorana spinors have $2^{\frac{[11]}{2}} = 2^{\frac{10}{2}} = 32$ degrees of freedom in 11 dimensions. Now assume that the $11D$ spacetime has the product structure $M_4 \times T^7$, where M_4 is a $4D$-spacetime and T^7 the torus in 7 dimensions. Products of this kind are the general ansatz for reducing the dimensionality of the theory in $D \geq 4$, but this will be seen in detail in the following section. Following this factoring of the 11-dimensional manifold, one may define a basis for the $11D$ γ-matrices, Γ^I as

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \times \mathbb{1}_8 \\ \Gamma^i &= \gamma_* \times \hat{\gamma}^i, \end{aligned} \tag{4.6}$$

where γ are the 4×4 matrices and $\hat{\gamma}$ are the $2^{\frac{[7]}{2}} = 8\times8$ matrices. Following the properties outlined before, these sets of matrices mutually anti-commute, and crucially, due to the presence of γ_*, the only non-trivial anti-commutation relations are between the Γ^μ's and the Γ^i's, respectively. One may wonder why $\mathbb{1}_8$ has been used rather than $\hat{\gamma}_*$: this is because for odd dimensions $\hat{\gamma}_*$ is, up to a phase factor, just the unit matrix [1]. Viewed now as a $4D$ theory, the Majorana gravitino can be indexed appropriately as is interpreted as $\Psi_{I\alpha b}$, where I is the initial spacetime index, α is the reduced spacetime and b is the $D = 7$ Majorana index. Under $4D$ Lorentz rotations, the $\Psi_{\mu\alpha b}$ sector transforms as $a \in \{1, ..., 8\}$ gravitinos and the $\Psi_{i\alpha b}$ transforms as $(i \in \{4, ..., 11\}) \times (b \in \{1, ..., 8\} = 56)$ spin-$\frac{1}{2}$ particles.

We now take pause to recall some facts about the particle content of supersymmetry representations. In particular, the maximum supersymmetries in $D = 4$ is $\mathcal{N} = 8$. To see this, take the limiting maximal helicty, $h = 2$. The most amount of half-steps from 2 to -2 is 8. No more supersymmetry can be squeezed into such a representation without taking $h > 2$, which we have already stated is not physical. Further more, 8 spin-$\frac{3}{2}$ particles and 56 spin-$\frac{1}{2}$ particles is the fermion content of this unique SUSY multiplet.

Therefore, the *minimal* ($\mathcal{N} = 1$) theory of supergravity in $D = 11$ corresponds to the *maximal* SUSY in $D = 4$. Increasing the supersymmetry in $D = 11$ or going to $D > 11$ would necessarily exceed the maximum SUSY in the $4D$ theory, therefore $D = 11$ is the maximum allowed dimension for a theory of supergravity.

4.2 D=11 supergravity Lagrangian

The previous argument behind $D = 11$ as the maximum dimension for supergravity gives only the fermionic content of the representation. One must introduce bosonic content into the theory to build up the whole SUSY representation and Lagrangian. As this is a gravity theory we can posit the existence of a vielbein field stemming from the spacetime metric, which is a massless spin-2 paritcle and therefore has $\frac{(D-2)(D-2+1)}{2} - 1 \to 44$ bosonic degrees of freedom. The Majorana spinor gravitino has $(D-3)2^{\frac{[D]}{2}} \to 256$ degrees of freedom, corresponding to 128 fermions. However supersymmetry requires equal numbers of bosons and fermions meaning we are $128 - 44 = 84$ short. The additional bosonic components are a point of great interest from the view of the mathematically rigorous formulation of supergravity. Although one can simply hypothesise the extra bosonic field as the 3-form gauge field $A_{\mu\rho\sigma}$ which correctly has $\binom{D-2}{3} = 84$ components, which was the case of the original conception of the $D = 11$ supergravity Lagrangian by Cremmer, Julia and Sherk [2], this is apparently ad hoc from the gauge theory perspective we have taken up until now. What is the gauge-theoretic understanding of $A_{\mu\rho\sigma}$? The graviton and gravitino were derived from the super-Cartan connection, but there was no mention of a 3-form gauge field. It is inelegant to have super-geometric formulation of supergravity which does not account for the complete field content of the theory and suggests it is not the correct perspective one should take. The correct interpretation was initiated by d'Auria and Fré [7], who generalised the Cartan geometry by introducing the concept of a *Cartan integrable system*. Briefly, a Cartan integrable system is premised on a generalisation of the Maurer-Cartan equation as a condition for local integrability of 1-forms to account for integration of p-forms which are, in turn, formulated using the original 1-form gauge fields. Such systems are described by *free differential algebras*, however this is beyond the remit of this paper. This concept was subsequently further developed in terms of *higher Cartan geometry* which is to p-form gauge fields as Cartan geoemtry is to 1-form gauge fields. Thus, super-Cartan geometry is encoded in $A_{\mu\rho\sigma}$ and the resulting formulation and notion of integrability is described by higher Cartan geometry.

We mention higher Cartan geometry just to acknowledge the consistency of the super-Cartan interpretation of supergravity which we have so far committed to employing. The important point is that the additional bosonic components are contained in the 3-form $A_{\mu\rho\sigma}$. Therefore, the complete field content of $D = 11$ supergravity is a graviton field, e, gravitino, Ψ_M, and *C-field* $A_{\mu\rho\sigma}$. The next step is to construct a Lagrangian from these components which is invariant under the symmetries of supergravity. Given we have introduced a 3-form, we can add a typical kinetic term 4-form field strength to the universal action

$$S = \frac{1}{2\kappa^2} \int d^{11}x\, e \Big[e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \Big]. \tag{4.7}$$

This is not the complete action however, as we must ensure that it is in fact supersymmetric. For this to be an invariant involves adding a term proportional to the supercurrent $\mathcal{J}^\nu$

$$\mathcal{J}^\nu = (\gamma^{\alpha\beta\sigma\delta\nu\rho} F_{\alpha\beta\sigma\delta} + 12\gamma^{\alpha\beta} F_{\alpha\beta}{}^{\nu\rho})\psi_\rho \tag{4.8}$$

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int d^{11}x\, e \Big[& e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \\ & - \frac{\sqrt{2}}{96} \bar{\psi}_\nu (\gamma^{\alpha\beta\sigma\delta\nu\rho} F_{\alpha\beta\sigma\delta} + 12\gamma^{\alpha\beta} F_{\alpha\beta}{}^{\nu\rho})\psi_\rho \Big], \end{aligned} \tag{4.9}$$

as is typical for interaction terms in the construction of gauge-field Lagrangians. The final term one must add is the so-called *Chern-Simons* term. It is related to cohomology of the underlying principal bundle which is extracted by the closed-form nature of F. The implication is that F also contributes as a topolgical term and introducing the Chern-Simons term counteracts this effect to restore supersymmetry [29, 1].

$$S_{CS} = -\frac{\sqrt{2}}{6\kappa^2} \int F^{(4)} \wedge F^{(4)} \wedge A^{(3)} \tag{4.10}$$

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int d^{11}x\, e \Big[& e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \\ & - \frac{\sqrt{2}}{96} \bar{\psi}_\nu (\gamma^{\alpha\beta\sigma\delta\nu\rho} F_{\alpha\beta\sigma\delta} + 12\gamma^{\alpha\beta} F_{\alpha\beta}{}^{\nu\rho})\psi_\rho - \frac{\sqrt{2}}{3} F^{(4)} \wedge F^{(4)} \wedge A^{(3)} \Big]. \end{aligned} \tag{4.11}$$

The Lagrangian (4.11) is consists of the key components of the $D = 11$ supergravity Lagrangian. However, some of the terms still require refinement owing to the principal of covariance for the field equations [2] (i.e. ensuring the form of the equations of motions is invariant under supersymmetry). Ensuring that the field equations are supercovariant results in the following final form of the 11-dimensional supergravity Lagrangian

$$\begin{aligned} S = \frac{1}{2\kappa^2} \int d^{11}x\, e \Big[& e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu^{(\frac{1}{2}(\omega+\hat{\omega}))} \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \\ & - \frac{\sqrt{2}}{96} \bar{\psi}_\nu (\gamma^{\alpha\beta\sigma\delta\nu\rho} + 12\gamma^{\alpha\beta} g^{\sigma\nu} g^{\delta\rho}) \psi_\rho (F_{\alpha\beta\sigma\delta} + \hat{F}_{\alpha\beta\sigma\delta}) \\ & - \frac{\sqrt{2}}{3} F^{(4)} \wedge F^{(4)} \wedge A^{(3)} \Big]. \end{aligned} \tag{4.12}$$

Here, $\hat{\omega}$ and $\hat{F}$ are supercovariant equivalents of the tensorial ω and F. They enter into the Lagrangian such that the field equations are dependent on them rather than their tensors counterparts. Fuller detatils and the explicit derivations of each of these terms can be found in the original paper presenting the $D = 11$ Lagrangian by Cremmer, Julia and Sherk [2].

To summarise, we have introduced the supergravity Lagrangian in 11 dimensions in the spirit of its original conception in [2]. Here, the authors deduce the Lagrangian by demanding its supersymmetry as well as the supercovariance of its field equations. We also briefly mentioned the more modern, geometrical understanding in terms of higher super-Cartan geometry [7, 30]. This description closes the understanding of supergravity to purely geometric, relating to an underlying super-Cartan geometry (such as that which describes $D = 4$, $\mathcal{N} = 1$, equation (3.35)). Using this recipe book, the Lagrangian can be understood as containing terms all analogous to each other through the concept of a generalised Maurer-Cartan equation (indeed these analogies or *equivalences* are part of how the ∞-group obejct of higher-Cartan geometry is defined [31]) as derivations of a higher super-Cartan geometry, all relating to the original super-Cartan geometry.

Not only is supergravity in $D = 11$ interesting for the fact that it is the maximum admissible dimension for such a theory, it also presents some brilliant results with regards to unification when the theory is reduced to 4 dimensions, the techniques of which we analyse in the following section. Indeed, the development of these techniques ushered in the understanding of higher dimensional theories as more than just mathematical curios but possibly containing the physical information of gauge symmetries when viewed in the lower dimension [32]. We have already shown how $D = 11$, $\mathcal{N} = 1$ supergravity is related to $D = 4$, $\mathcal{N} = 8$ supersymmetry in justifying $D = 11$ being maximal. As is the theme of this paper, we first introduce these techniques as they originally applied to ordinary gravity. We follow this up by the supergravity analogue which, as ever, was inspired by its gravitiational predecessor.

Chapter 5

Lowering Dimensions

In this chapter we discuss methods of dimensional reduction[1] of physical theories. This is integral to any theory formulated in dimensions higher than those observed in the real world, currently four. For example, to reconcile $D = 10$ superstring theory and $D = 11$ supergravity theories with the $D = 4$ physics of general relativity, there must exist tools for formally reducing the higher-dimensions [32]. The idea of dimensional reduction of physical theories can be traced back to the original Kaluza-Klein theory [33, 34], and is where the general process gets its name, which unifies electromagnetism and gravity and originated as a theory of general relativity in $D = 5$ ($D = 5$ spacetime with diffeomorphism invariance and LL symmetry). Gravity as observed is then restored by compacting the extra dimension as S^1 such that all fields are independent of this extra dimension [35]. Thus, this is a dimensional reduction from D to $D-1$ dimensions; KK-reduction is of this nature in general, where higher dimensions are compactified as manifolds and higher order massive modes resulting from the compactness of the extra dimensions are dropped, providing that the field equations of the reduced theory do not source the higher dimensional modes. That is, the lower dimensional theory offers a self-consistent set of field equations. This section introduces Kaluza-Klein methodologies following the same chronology as they were developed.

5.1 Kaluza-Klein theory

The starting point for Kaluza-Klein theory is Einstein's gravity in $D > 4$. This began with Kaluza [33] who generalised general relativity to $D = 5$. When we talk of generalising gravity to higher dimensions what we really mean is gauge theories with diffeomorphisms of spacetime. As we shall see these spacetime symmetries will be transformed into gauge symmetries. This was half the point of section 2, to introduce the principles of gauge theory, as well as displaying gravity as a specific case. This was also to draw comparisons to the gauge theories of the standard model and beyond (i.e. supergravity) which will be needed when we consider dimensional reductions of supergravity theories. The separate streams of gravity and other gauge theories will hence begin to converge in the philosophies which developed in Kaluza-Klein theory.

Having accepted $D > 4$ gravity, the next step in KK-theory is to postulate the existence of a vacuum product space, $M_4 \times M_k$, as a solution to the field equations of the higher dimensional theory [32, 36]. We will denote the coordinates of this product space by (x^μ, y^a), where Greek and Roman indices pertain to the $D = 4$ and compact spaces, respectively; capitalised indices such as $M \in \{0, 1, ..., D\}$ are total space indices. As half the product is a compact submanifold, the components of any field excitation about such a vacuum can be correspondingly separated into a part describing the $D = 4$ theory multiplied by a part which is a function of the compact space; fields over a compact space then admit expansions in terms of the orthogonal harmonic modes of

[1]A note on potential confusion. There are two cases where we use the term 'dimensional reduction'. First as a generic description of any process that reduces the dimensionality of a theory, which we will refer to as normal, and dimensional reduction as a specific mechanism which ignores any dependence on external coordinates. We will attempt to be clear which we are refering to.

the space [32, 37]

$$\hat{g}_{MN}(x,y) = \langle \hat{g}_{MN}(x,y) \rangle + \sum_{n=-\infty}^{\infty} \Big(g(x) Y_g(y) \Big)_{MN} \tag{5.1}$$

$$\hat{\Phi}_{MNK\ldots}(x,y) = \langle \hat{\Phi}_{MNK\ldots}(x,y) \rangle + \sum_{n=-\infty}^{\infty} \Big(\Phi(x) Y_\Phi(y) \Big)_{MNK\ldots} \tag{5.2}$$

Here, the fields over x and over y contribute either Greek or Roman indices to the tensor, respectively. Equation 5.1 is the expansion about the vacuum value of the metric tensor, given by the product space tensor

$$\langle \hat{g}_{MN}(x,y) \rangle = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix}, \tag{5.3}$$

and equation 5.2 refers to the mode expansion around the vacuum of a generic tensor field defined over the manifold. The functions over y are spherical harmonic functions, i.e. the satisfy the mass equation given by the Laplace operator over the sphere, Δ, equipped with the standard metric in analogy with the derivation of scalar spherical harmonics over S^2 [38]

$$\Delta Y_{(n)} = m_n^2 Y_{(n)}. \tag{5.4}$$

As this expansion is parameterised by the extra compact dimensions, to touch base with the real world physics of $d = 4$, at least at low energy; the energies of everyday experience, it should be required that these expansions are truncated to all but trivial dependence on y^a - the so-called *cylinder condition* [32]. This births the so-called "Kaluza-Klein Ansatz" for the form of the metric. If a metric does not depend on a certain coordinate, then flows along that coordinate are generated by a *Killing vector field*. The set isometries of the metric produced by the Killing vector fields also form a Lie group, thus a manifold with Killing vector fields can be viewed as parameterising a group action. This connection is how we will see gauge theory emerge over the submanifold M_4. Therefore, the Kaluza-Klein ansatz for the metric tensor is parameterised by Killing vectors, so for low energies

$$\begin{aligned} \hat{g}_{\mu\nu}(x,y) &= g_{\mu\nu}(x,y) + A_\mu{}^\alpha(x) A_\nu{}^\beta(x) K^{m\alpha}(y) K^{n\beta}(y) \tilde{g}_{mn}(y) \\ \hat{g}_{\mu n}(x,y) &= A_\mu{}^\alpha(x) K^{m\alpha}(y) \tilde{g}_{mn}(y) \\ \hat{g}_{mn}(x,y) &= \tilde{g}_{mn}(y), \end{aligned} \tag{5.5}$$

where the indices run over the ranges specified before. We see explicitly here that the only excitations are those for which $\tilde{g}_{mn}(y)$ is invariant. This ansatz is further justified by substituting (5.5) into the action of the higher dimensional theory and, because we have this assumes no y-dependence, we can integrate over y to get an effective Einstein-Yang-Mills theory over the $4D$ submanifold. As well-reasoned as this ansatz may seem, in general it is broken due to inconsistency with the higher-dimensional action. The first instance of this ansatz as developed by Kaluza and Klein [33] for gravity in $D = 5$, however, is the exception and is referred to as the Kaluza-Klein miracle.

5.1.1 D=5 Kaluza-Klein theory

Starting with an Einstein-Hilbert-like action in $D = 5$ with local coordinates (x^μ, y)

$$S = \frac{m}{2\pi\kappa^2} \int d^4x dy \sqrt{-g} R_5. \tag{5.6}$$

Owing to the principal of general covariance of the field equations, this action is invariant under general coordinate transformations

$$\delta \hat{g}_{MN} = \partial_M \xi^P \hat{g}_{PN} + \partial_N \xi^P \hat{g}_{PM} + \xi^P \partial_P \hat{g}_{MN}. \tag{5.7}$$

Making the assumption that the ground state space is the product manifold $M_4 \times S^1$, we reformulate the metric $\hat{g}$ inspired by the required low energy form (5.5) as

$$\hat{g}_{MN} = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} + \kappa^2 \phi A_\mu A_\nu & \kappa\phi A_\mu \\ \kappa\phi A_\nu & \phi. \end{pmatrix} \tag{5.8}$$

where the resemblance[2] to (5.5) emphasises that it should be the low energy limit of(5.8). We now generically expand $g_{\mu\nu}$, A_μ, and ϕ in the form of (5.2) and insert these into (5.8)

$$\phi(x,\theta) = \sum_{n=-\infty}^{\infty} \phi_n(x) e^{in\theta} \quad A_\mu(x,\theta) = \sum_{n=-\infty}^{\infty} A_{\mu n}(x) e^{in\theta} \quad g_{\mu\nu}(x,\theta) = \sum_{n=-\infty}^{\infty} g_{\mu\nu n}(x) e^{in\theta} \tag{5.9}$$

We then formally make the KK ansatz by truncating the expansions for $n > 0$, thus eliminating all y-dependence. Substituting this into (5.6) and integrating over y, which is now taken to be a periodic coordinate, $my = \theta$; $0 \le \theta \le 2\pi$, yields the effective $4D$ action

$$S = \frac{1}{2\pi\kappa^2}\int_0^{2\pi} d\theta \int d^4x \sqrt{-\hat{g}}\hat{R}_5 = \int d^4x \sqrt{-g_4}\Big(\frac{R_4}{\kappa^2} - \frac{1}{4}\phi F_{\mu\nu}F^{\mu\nu} - \frac{1}{6\kappa^2\phi^2}\partial^\mu\phi\partial_\mu\phi\Big). \tag{5.10}$$

With the transformation of the $5D$ action (5.6) to the $4D$ effective action (5.10), the diffeomorphism invariance (5.7) is converted to the following symmetries

$$\begin{aligned} \delta g_{\mu\nu} &= \partial_\mu\xi^\rho g_{\rho\nu} + \partial_\nu\xi^\rho g_{\rho\mu} + \xi^\rho\partial_\rho g_{\mu\nu} \\ \delta A_\mu &= \partial_\mu\xi^\rho A_\nu + \xi^\rho\partial_\rho A_\mu \\ \delta\phi &= \xi^\rho\partial_\rho\phi \end{aligned} \tag{5.11}$$

for $4D$ spacetime index $\rho \in \{0, ..., 3\}$ and

$$\delta A_\mu = \frac{1}{\kappa}\partial_\mu\xi^4, \tag{5.12}$$

where ξ^4 is the coordinate of the 5^{th} dimension. We see clearly that (5.11) are the diffeomorphisms of M_4, intrinsic to GR, and (5.12) is exactly of the form of a gauge transformation. Thus, the symmetries include[3] general covariance of GR and gauge invariance of electromagnetism; we have a gauge theory! A natural next question to ask is: how is the ansatz justified physically? For this we look to the mass spectrum of by restoring the $n \neq 0$ modes. Still using the split $M_4 \times S^1$, we apply the $5D$ Klein-Gordon equation to a generic field or field component [39]

$$\begin{aligned} \Box\phi(x,\theta) &= \Big(\frac{\partial}{\partial x^2} + \frac{1}{R^2}\frac{\partial}{\partial\theta^2}\Big)\sum_{n=-\infty}^{\infty}\varphi_n(x)e^{in\theta} = 0 \\ &= \sum_{n=-\infty}^{\infty}\Big(\Box_x\varphi_n(x) - \frac{n^2}{R^2}\varphi_n(x)\Big)e^{in\theta} = 0. \end{aligned} \tag{5.13}$$

The extra dimensions are taken to have spacelike signature, to avoid tachyons [32]. This leads to a massive Klein-Gordon equation for the x-dependent components with masses m_n

$$\begin{aligned} \Box_x\varphi_n(x) - \frac{n^2}{R^2}\varphi_n(x) = 0 \\ m_n^2 = \frac{n^2}{R^2}. \end{aligned} \tag{5.14}$$

Thus, these masses decouple from the massless mode ($n = 0$) when $R \to 0$. Not only can the massive modes which correspond to non-trivial excitations in the y direction be relegated to high energy regimes, the scenario for which this happens corresponds to an infinitesimal extra dimension which fits in nicely with ones lack of observation of it.

What is 'miraculous' about (5.10) is that its field equations are completely consistent - they are not sources of the higher mass terms; having field equations that produce the higher order terms we excluded in our ansatz is obviously inconsistent. However, this is not true in general and the ansatz (5.5) is not consistent, as was demonstrated by Duff et al. [40]. This issue of inconsistency is not present when one retains all massive modes, but then one is left with the job of interpreting what the total theory means physically.

[2]The scaling factor in front of the matrix is to ensure that the field equations yield the canonical form of gravity [39].

[3]There is also a scale invariance $\delta A_\mu = \lambda A_\mu, \delta\phi = -2\lambda\phi.$, facilitated by the presence of the dilaton, ϕ. However, we restrict to the heuristic arguments of the KK theory being a possible tool for unified theories; the dilaton represents new physics.

5.1.2 Kaluza-Klein theory in 4+D dimensions

With the original and most basic example of $5D$ KK theory out of the way, we now introduce the higher dimensional generalisation meant to include non-Abelian gauge groups. Given the general idea of KK reductions to posit extra dimensions as manifolds that parameterise a group manifold of interest, the obvious choice of extending this process would be, for example for SU(3), to factor the manifold S^3 into the product manifold. We thus layout the general prescription of KK theory for any dimension as follows. We start by assuming a split $M_4 \times M_{D-4}$ with M_{D-4} compact and apply the Kaluza-Klein ansatz (5.5)

$$\begin{pmatrix} g_{\mu\nu} + A_\mu{}^\alpha A_\nu{}^\beta K^{m\alpha} K^{n\beta} \tilde{g}_{mn} & A_\mu{}^\alpha K^{m\alpha} \tilde{g}_{mn} \\ A_\mu{}^\alpha K^{m\alpha} \tilde{g}_{mn} & \tilde{g}_{mn} \end{pmatrix}. \tag{5.15}$$

With some foresight, we now take the following diffeomorphism on the compact submanifold

$$y'^m = y^m + \sum_{i=1}^{n} \epsilon^\alpha(x) K_\alpha{}^m(y), \tag{5.16}$$

where we recall that the $K_i{}^m$ are the n Killing vectors of the compact space. The Kaluza-Klein ansatz has restricted us to the lowest order fluctuations of the fields, such that they have no dependence on y - hence the Killing vector parameterisation. By this definition, the metric $\tilde{g}$ is invariant under such diffeomorphisms, however other components of the metric do change. Using (5.7) one can deduce that the field $A_\mu{}^\alpha$ changes in the following familiar way [32, 39]

$$A_\mu{}^\alpha(x) \to A_\mu{}^\alpha(x) + \partial_\mu \epsilon^\alpha(x) - f^i{}_{jk} A_\mu{}^j(x) \epsilon^k(x), \tag{5.17}$$

where $f^i{}_{jk}$ are the structure constants of the Lie algebra generated[4] by the Killing vectors, $K_i{}^m$. Thus, we once again see that the diffeomorphisms of the higher-dimensional theory have resulted in gauge symmetries over the lower-dimensional space. However, as we have previously alluded to, everything is not as promising as it seems from this result; the ansatz is not consistent. An easy way of seeing this as shown in [40] is to add a cosmological constant term to the pure gravity Lagrangian (5.6), which admits the desired split $M_4 \times M_{D-4}$ with M_{D-4} compact as a solution of the field equations,

$$R_{MN} = \Lambda g_{MN}. \tag{5.18}$$

Substituting the ansatz (5.15) into (5.18) gives

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{1}{2}\Big(F_{\mu\rho}{}^\alpha F_\nu{}^{\rho\beta} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}{}^\alpha F^{\rho\sigma\beta} \Big) K_{n\alpha} K^n{}_\beta. \tag{5.19}$$

The inconsistency can be seen by comparing the left and right-hand sides of this equation. On the left we have an equation defined strictly over M_4, while on the right we have Killing vectors, $K_n{}^\alpha$, which are (in general) inherently dependent on y. So, despite obtaining the desired splitting of manifolds, what works for the lower dimensional theory does not work for the higher dimensional theory and so they cannot be consistently related. Furthermore, in the absence of a cosmological constant, there is no solution to the field equations which gives the desired splitting, so the ansatz is broken. We are thus presented with two solutions to restore consistency: we can include all massive modes and drop the ansatz for which issues of consistency are trivial, or find cases for which $K_{n\alpha} K^n{}_\beta = \delta_{\alpha\beta}$ which removes the y-dependence. With regards to the first option, as to why ones only option is to include *all* massive modes, not truncate at some finite number, it was shown by Duff, Pope and Stelle [41] that including all states is the only consistent choice. Here, they used the case of $D = 5$ as an exemplar. Accepting a truncation of $n \neq 0$ allows for the introduction of general diffeomorphisms over the extra dimension which follow the splitting (5.2)

$$\xi^M(x, \theta) = \sum_{n=-\infty}^{\infty} \xi_n^M e^{in\theta}. \tag{5.20}$$

These diffeomorphisms thus correspond to a 'tower' of symmetries. If one takes all the scalar fields offered in the expansion (5.9) then the changes under (5.20) cause a change to a level n of

$$\delta\phi_n(x) \sim \sum_r \epsilon_r \phi_{n-r}, \tag{5.21}$$

[4]It is a generic property of Killing vectors generating isometries (symmetry groups) of a metric space that they form a Lie algebra (of the relevant symmetry group).

where these are all contributions that yield the same wavenumber in the periodic component of ϕ_n and ϵ_n is a the n^{th} parameter. Thus the fields ϕ_n do not in general close under subset of (5.20). This can be seen explicitly by considering that the subset ξ^5_{-1}, ξ^5_0, ξ^5_1 generates the *non-compact subgroup*[5] SO(2,1). As is well known, any unitary representation of a non-compact group must be infinite-dimensional. Hence the form of (5.21) and why a finite truncation is not consistent.

The other option we have is to find subgroup, G', with associated manifold such that

$$K_{n\alpha}K^{n}{}_{\beta} = \delta_{\alpha\beta} \tag{5.22}$$

is true [40]. Being compact, the subgroup manifold M'_k admits a bi-invariant metric which, when chosen, thus yields $G' \times G'$ as its full group of isometries (by definition of its bi-invariance). Clearly, choosing Killing vector fields which are either left or right-invariant (corresponding to the left or right group action in the product) are by our initial assumption Killing vectors which obey (5.22). However this is not true in general for Killing vectors which correspond to the cross-terms of the product, and can, in fact, spring-up unwanted dependence on the extra dimensions. Therefore only the subgroup, G', provides a consistent set of isometries. The tower of massive modes then facilitates the full set of isometries. This type of dimensional reduction was first introduced by deWitt [42]. Explicitly, for this to be consistent, the only fields retained after the truncation must be singlets under one of groups, i.e. the truncated modes are in non-trivial representations of that gauge group [43].

This alludes to a group-theoretic interpretation of both the KK ansatz and the consistency condition. We rephrase our understanding of using isometries of the compact manifold to foster independence of the extra dimensions as needing the low-energy, truncated theory to be comprised of singlets of the corresponding group. We take the low-energy manifold to be a submanifold of some larger Lie group, parameterised by the non-trivial group representations which are truncated. This more general interpretation of the KK ansatz is referred to as the *K-invariant ansatz*, where K is the subgroup in question [44]. Moreover, this interpretation gives an alternative view into consistency. Firstly, no linear terms of the truncated fields, H, are allowed for a consistent theory. If there were, then, for singlet fields, ϕ, and H-linear interaction term, $\lambda H\phi^2$, the equations of motion for H would be of typical form

$$(\nabla^M\nabla_M + m^2)H + \lambda\phi^2 = 0. \tag{5.23}$$

But this is clearly not allowed as it is not tensorial [41]. So, with regards to consistency, setting $H = 0$ is now a valid option as a means of truncation which satisfies the field equations of H and thus we have consistency by way of eliminating non-trivial representations of the subgroup, K. On this note, in general it is required that we retain *all* singlets, ϕ, as there is no group-theoretic argument that precludes them from acting as sources for each other [44, 43].

This concludes our discussion on non-supersymmetric Kaluza-Klein theory. The many hurdles that one has to jump over to force such KK theories to work suggests that this is not the correct way for unifying the particles of the standard model. However, this is no longer necessarily the case when one considers supergroups and it is believed that KK theory is the tool-of-the-trade for higher dimensional supergravity , specifically $D = 11$ [32].

5.2 Kaluza-Klein and supergravity

We already alluded to the relationship[6] between higher and lower-dimesnional theories of supergravity when we introduced in section 4.1, where we saw that $\mathcal{N} = 1$, $D = 11$ supergravity yields the maximally supersymmetric $\mathcal{N} = 8$, $D = 4$ theory. Having such an upper bound on the possible theories of supergravity, provides a security from the Kaluza-Klein perspective: one has a definite theory with a definite Lagrangian with which to work with, there are now no concerns about what higher dimensional theory one ought to consider; there is now only one toy in the toybox.

We thus now turn our attention to the supersymmeterised sibling on the Kaluza-Klein mechanism. As one might imagine the morals are similar in both the SUSY case as in the non-SUSY case, but where in the former we now take supergroups into consideration.

[5]The entire algebra formed from 5.20 is known as the Kač-Moody extension to the Poincaré algebra. Further details on this can be seen in e.g. [32].

[6]Note, however, the higher-dimensional fields are independent of the extra coordinates and are just reinterpreted as higher-rank tensors over the 4D manifold - this is an example of a dimensional reduction of the theory.

5.2.1 The Freund-Rubin ansatz

The Freund-Rubin (FR) ansatz is a dimensional reduction method of the $D = 11$ supergravity Lagrangian. It is not a Kaluza-Klein method where one considers a tower of massive states, but rather demands in the ansatz that the field content is independent of the extra dimensions. We first remind ourselves of the $D = 11$ supergravity Lagrangian (4.12), with a few amendments to notation to be consistent with the conventions of this chapter

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^{11}x\, e\Big[&e^{AM}e^{BN}R_{MNAB}(\omega) - \bar{\Psi}_M\gamma^{MNP}D_N^{(\frac{1}{2}(\omega+\hat{\omega}))}\Psi_P - \frac{1}{24}F^{MNPQ}F_{MNPQ} \\ &- \frac{\sqrt{2}}{96}\bar{\Psi}_N(\gamma^{MNPQRS} + 12\gamma^{MN}g^{PR}g^{QS})\Psi_S(F_{MPQR} + \hat{F}_{MPQR}) \\ &- \frac{\sqrt{2}}{3}F^{(4)}\wedge F^{(4)}\wedge A^{(3)}\Big]. \end{aligned} \tag{5.24}$$

The ansatz then asserts the typical splitting with which we are now more than familiar $M_4 \times M_7$, and we search for solutions of fields over M_4 which are independent of M_7 and consistent with the higher-dimensional field equations. One makes the general assumption that the lower-dimensional spacetime should be maximally symmetric, which forces the Vacuum Expectation Values (VEVs) of all fields to be comprised of Lorentz-invariant quantities. For the gravitino field, Ψ_M, this implies it should have vanishing VEV as this is the only invariant spinor under the Lorentz action and also, on the group-theoretic grounds of the preceding section, this is a valid solution of the original field equations as there are no terms linear in Ψ_M. The field equations for the remaining fields are then

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{3}\big[F_{MPQR}F_N{}^{PQR} - \frac{1}{8}g_{MN}F_{PQRS}F^{PQRS}\big] \tag{5.25}$$

$$\nabla_M F^{MNPQ} = -\frac{1}{576}\epsilon^{M_1\dots M_8 PQR}F_{M_1\dots M_4}F_{M_5\dots M_8}. \tag{5.26}$$

Explicit derivations can be found in A.2. We thence look for solutions to these equations that allow for maximal spacetime symmetry and follow the splitting $M_4 \times M_7$. The conclusion is that the ansatz should be of the structure

$$\begin{aligned} &\langle g_{\mu\nu}\rangle = \mathring{g}_{\mu\nu}(x), \quad \langle F_{\mu\nu\sigma\rho}\rangle = \mathring{F}_{\mu\nu\sigma\rho}(x), \quad \langle F_{\mu\nu\rho q}\rangle = 0 \\ &\langle g_{mn}\rangle = \mathring{g}_{mn}(y), \quad \langle F_{mnpq}\rangle = \mathring{F}_{mnpq}(y), \quad \langle F_{\mu\nu pq}\rangle = 0, \\ &\langle g_{\mu n}\rangle = 0 \qquad\qquad \langle \Phi_M\rangle = 0 \qquad\qquad \langle F_{\mu npq}\rangle = 0 \end{aligned} \tag{5.27}$$

which posits that all fields can be cleanly separated. One then takes

$$\begin{aligned} \mathring{F}_{\mu\nu\sigma\rho} &= 3m\sqrt{-\mathring{g}}\epsilon_{\mu\nu\sigma\rho} \\ \mathring{F}_{mnpq} &= 0, \end{aligned} \tag{5.28}$$

with m constant. Equation (5.28) is then the *Freund-Rubin ansatz* [32, 4]. It is clear how the ansatz satisfies (5.26), while substituting it into (5.25) yields a seperation into two curvature tensors

$$\mathring{R}_{\mu\nu} = -12m^2\mathring{g}_{\mu\nu} \qquad \mathring{R}_{mn} = 6m^2\mathring{g}_{mn}, \tag{5.29}$$

where we take the latter to have signature $(-+++)$ and the former to have $(+++++++)$ based on the initial theory's SO(1,10) invariance with signature for g_{MN}, $(-++++++++++)$. We note that a Riemannian geometry with everywhere postive curvature is compact [45]. Then, given the presence of m^2, one makes the discovery that the curvatures correspond to the maximally symmetric spacetime of AdS$\times M_7$, with M_7 compact - just as required! There are few comments to make based on the original paper [4]. Firstly, the split of the larger space into 4 and 7- dimensional submanifolds is much more than an ad hoc choice that we wanted to enforce. The field content of $D = 11$ contains the gauge field A_{MNP} and thus potential term F_{MPQR} necessarily of rank 4. When searching for invariant tensors in the lower dimension, the *only* option one can take for the epsilon-tensor is the one of rank 4 if it is to also satisfy the field equations. Secondly, the derivation also allows for the choice of signatures $(++++)$ and $(-++++++)$. We exclude this option from our discussion on the basis that it is less desirable from a phenomenological point of view.

This defines the basic components of the Freund-Rubin ansatz. Strictly speaking, we have a compactified solution which has sepearated into two submanifolds, however this is the ripe start one needs, to apply a Kaluza-Klein mechanism and yield the desired result of physics over a 4-dimensional manifold. We now delve deeper into the result of Anti-de Sitter$_4 \times M_7$ (AdS$_4 \times M_7$). We now begin to consider preserved supersymmetry on the lower dimension, where our understanding of supermanifolds from chapter 3 will prove useful.

5.2.2 AdS spacetime and Killing spinors

Anti-de Sitter spacetime is the maximally symmetric spacetime with negative Ricci scalar. Following our discussion of Klein and Cartan geometry of chapter 2, we declare that AdS$_4$ is the Klein geometry (SO(2, 3), SO$^+$(1, 3)) and so the space as a whole is invariant under SO(2, 3) but stabilised by SO$^+$(1, 3). Importantly, AdS admits a spin structure, allowing one to define fermionic fields. This therefore allows for the superymmetric extension of its bosonic symmetry group - the *orthosymplectic* super-Lie group OSp($4|\mathcal{N}$), where $\mathcal{N}$ can be any number of allowed supercharges $0 \leq \mathcal{N} \leq 8$ [32]. Indeed, if we desire the vacuum solutions (5.27) to retain supersymmetry one requires that the condition $\langle \Phi_M \rangle = 0$ is supersymmetrically invariant. This requires *Killing spinors* which can be understood by recasting the fields as over a supermanifold.

Recall that we introduced the gravitino, Φ_M, as the Grassmann-odd part of the supervielbein for a super-Cartan geometry. That is, it provides a map between elements of the super-coset space of the underlying super-Klein geometry and the tangent space of the bosonic submanifold. Now, consider that a choice of vielbein, e, over a regular Cartan geometry induces a metric, g on the spacetime manifold through the metric over the Cartan geometry, η, via

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}, \tag{5.30}$$

where Roman indices refer to the Lie algebra index, and Greek indices the spacetime index. In an analogous manner, the supervielbein induces a super-metric over the supermanifold. To see this, we first define a super-metric as follows

Definition 5.1 (Super-metric)**.** Take $\mathcal{V}$ to be a super-vector space, and $\Lambda^{\mathbb{C}} = \Lambda \otimes \mathbb{C}$ a $\mathbb{C}$-valued Grassmann algebra. Then a super-metric, $\mathcal{G}$, is a non-degenerate bilinear map: $\mathcal{V} \otimes \mathcal{V} \to \Lambda^{\mathbb{C}}$, itself with intrinsic parity $|\mathcal{G}| \in \mathbb{Z}_2$, such that

$$\mathcal{G}(v, w) = (-1)^{|v||w|}\mathcal{G}(w, v), \ \ \forall v, w \in \mathcal{V} \tag{5.31}$$

where $\mathcal{G}(\mathcal{V}_i, \mathcal{V}_j) \subseteq (\Lambda^{\mathbb{C}})_{|\mathcal{G}|+i+j}$.

Comments.

1. This is analogous to the function of an ordinary metric, except the image of the map under both its arguments is replaced by a 1D superspace, $\Lambda^{\mathbb{C}}$, rather than 1D Euclidean space.

2. The graded nature of this definition accounts for the graded nature of the vector space, $\mathcal{V}$.

Returning to the previous point about supervielbeins, we may define the induced super-metric, $\mathcal{G}$, over a super-spacetime in the spirit of (5.30)

$$\mathcal{G}_{IJ} = (-1)^{|i||j|}\mathcal{L}_{ij} E_I{}^i E_J{}^j, \tag{5.32}$$

where $\mathcal{L}$ is the metric over the super-Klein geometry, E is the supervielbein, and both indices run over the same dimensional superspace. With this definition in hand it is clearer to understand what a Killing spinor might be by comparing it to its bosonic counterpart, the Killing vector. Thus we define a Killing spinor as generating a flow over which supermetric does not change, and in particular this means that Ψ does not change. It can hence be shown [26] that, using this definition, a Killing spinor, ϵ, satisfies

$$\delta_\epsilon \Psi = D^{(\omega)}\epsilon - \frac{1}{2L} e^\mu \gamma_\mu \epsilon \equiv \tilde{D}^{(\omega)}\epsilon = 0, \tag{5.33}$$

which, unsurprisingly, is also the SUSY transformation rule for the gravitino [1] - we have supersymmetry! Following the AdS$\times M_7$ split, we search for solutions to (5.33) of the form

$$\epsilon(x, y) = \xi(x)\eta(y), \tag{5.34}$$

Along a similar vein to (4.6), we define a basis for the Γ-matrices as

$$\hat{\Gamma}_A = (\gamma_\alpha \otimes \mathbb{1}, \gamma_5 \otimes \Gamma_a), \tag{5.35}$$

where, again, γ_α are 4×4 and Γ_a are 8×8 matrices, and the corresponding metric signatures imply

$$\{\gamma_\alpha, \gamma_\beta\} = -2\eta_{\alpha\beta}, \qquad \{\Gamma_a, \Gamma_b\} = -2\delta_{ab}. \tag{5.36}$$

This causes the Killing spinor equation to decompose, yielding two separate sets of isometries

$$\tilde{D}^{(\omega)}_\mu \epsilon(x) = 0, \qquad \tilde{D}^{(\omega)}_m \eta(y) = 0 \tag{5.37}$$

The parameters ϵ, η are sections of the spin bundles of the submanifolds, as defined by the decomposition (5.35). The first condition in (5.37) is automatically satisfied by the global OSp symmetry of (super-)AdS$_4$, in the same way (super-)Minkowski space is an invariant space under the super-Poincare group action. Therefore, the residual supersymmetries are provided by the Killing spinors of M_7. Thus the size of our unbroken local supersymmetry is given by set of solutions to $\tilde{D}^{(\omega)}_m \eta(y) = 0$. Then, since the corresponding Clifford algebra is comprised of 8×8 matrices, η is an 8-component spinor, meaning we we have the possibility of $0 \leq \mathcal{N} \leq 8$. In addition to the conditions (5.37), we impose an *integrability condition* on the Killing spinors

$$[\tilde{D}_m, \tilde{D}_n]\eta = -\frac{1}{4} R_{mn}{}^{ab}\Gamma_{ab}\eta + \frac{1}{2} m^2 \Gamma_{mn}\eta. \tag{5.38}$$

An integrability condition is necessary as we ultimately dealing with actions which are integrals. It ensures that integrals involving the fields which the Killing spinor acts on are well defined. The above can be interpreted a supercovariant analogue of the Maurer-Cartan equation. The failure of either side of (5.38) to vanish implies that super-covariant derivatives D_m, D_n do not commute. This failure to commute is recorded by the linear combinations of the matrices $\Gamma_{ab} = \Gamma_{[a}\Gamma_{b]}$ and the subspace which they consequently generate is called the *holonomy* group of the connection in $\tilde{D}$ [46, 32]. The spinors that thus satisfy $-\frac{1}{4} R_{mn}{}^{ab}\Gamma_{ab}\eta + \frac{1}{2} m^2 \Gamma_{mn}\eta = 0$ define well-behaved Killing spinors and define the supersymmetry group of M_4. Now, the game to play is finding which spacetimes are admissible for M_7 and what physics they contain. However, we restrict our discussion to the interesting case of $M_7 = S^7$, the 7-sphere.

5.2.3 The 7-sphere

The 7-sphere is of particular interest as it is maximally symmetric in that it admits the maximum possible number of Killing vectors on a 7-dimensional space. To see this, assume that we have the maximum number of Killing spinors allowed by (5.37-5.38). In this case

$$[\tilde{D}_m, \tilde{D}_n]\eta = -\frac{1}{4} R_{mn}{}^{ab}\Gamma_{ab}\eta + \frac{1}{2} m^2 \Gamma_{mn}\eta = 0. \tag{5.39}$$

Rearranging the right-hand side sets the condition as

$$R_{mnpq} = m^2(g_{mp}g_{nq} - g_{mq}g_{np}), \tag{5.40}$$

which corresponds to a maximally symmetric space, i.e the 7-sphere. Since we have been dealing with cases of extreme symmetry, it is the natural choice to make. In addition, because of this, it produces many desirable results with regards to unification.

To briefly recap, we have introduced the Freund-Rubin compactification ansatz and shown how it permits a low-energy splitting of the underlying manifold into AdS$_4 \times M_7$, where M_7 is compact. We next considered introducing residual supersymmetry of this vacuum space. This lead to concept of a Killing spinor to maintain a supersymmetric vacuum (keep the vacuum supermanifold fixed) of the higher-dimensional theory. Following the splitting of the manifold, we factored the Killing vector into elements over AdS$_4$ and M_7 respectively, with reasonable conditions on integrability enforced. This defined the parameters of a respectable low-dimensional theory. Of the possible candidates for M_7 we selected the maximally supersymmetric choice of S^7. We now return to Kaluza-Klein and prepare the particle fields into regular functions over AdS$_4$ and harmonic functions over S^7, as per the Kaluza-Klein setup (5.1)-(5.4). We first consider the bosonic sector;

following (5.1) and (5.2) we infinitesimally expand the metric and field strength about their VEVs (5.28)

$$g_{MN}(x,y) = \mathring{g}_{MN}(x,y) + h_{MN}(x,y) \tag{5.41}$$

$$F_{MNPQ}(x,y) = \mathring{F}_{MNPQ}(x,y) + f_{MNPQ}(x,y). \tag{5.42}$$

Note that we are working with the field strength, F, rather than the gauge field, A. This is because the ground state of F as dictated by the FR ansatz is proportional to ϵ_{MNPQ}. While this is trivially a closed form, it is not exact - it cannot be written in terms of a gauge field. For this reason we also insert the expansions (5.41)-(5.42) into the *field equations*, (5.25)-(5.26), rather than the Lagrangian (5.24) since the Lagrangian explicitly involves both F and A. The caveat to this is that, to ensure that all expansions in F remain closed, we enforce the Bianchi identity that would otherwise automatically be satisfied by virtue of F being exact

$$\partial_{[M}F_{NPQR]} = 0. \tag{5.43}$$

With the particulars of the procedure considered, we can immediately substitute (5.41) and (5.42) into our field equations and 5.43 to give the low-energy field equations [37]

$$\begin{aligned}\delta R_{AB} &= \frac{1}{2}\hat{\Delta}h_{AB} + \nabla_{(A}\nabla^{C}h_{B)C} - \frac{1}{2}\nabla_A\nabla_B h^C{}_C \\ &= \frac{1}{3}\mathring{F}_{(A}{}^{CDE}f_{B)CDE} - \frac{1}{18}\mathring{g}_{AB}\mathring{F}_{CDEF}f^{CDEF} - \mathring{F}_{(A}{}^{CDE}\mathring{F}_{B)}{}^{C'}{}_{DE}\,h_{CC'} \\ &\quad - \frac{1}{36}h_{AB}\mathring{F}_{MNPQ}\mathring{F}^{MNPQ} + \frac{1}{9}\mathring{g}_{AB}\mathring{F}_{CDEF}\mathring{F}_{C'}{}^{DEF}h^{CC'}\end{aligned} \tag{5.44}$$

$$\begin{aligned}\nabla_M f^{MNPQ} &+ \frac{1}{2}\mathring{F}^{MNPQ}\nabla_M h^C{}_C - \mathring{F}_S{}^{NPM'}h^{MM'} - \mathring{F}^M{}_{N'}{}^{PQ}h^{NN'} \\ &- \mathring{F}^{MN}{}_{P'}{}^{Q}h^{PP'} - \mathring{F}^{MNP}{}_{Q'}h^{QQ'} = -\frac{1}{288}\epsilon^{M_1\ldots M_8NPQ}F_{M_5\ldots M_8}f_{M_1\ldots M_4}\end{aligned} \tag{5.45}$$

$$\nabla_{[M}f_{NPQR]} = 0, \tag{5.46}$$

where $\hat{\Delta}$ is the *Lichnerowicz operator* for the compact space, details of this and the corresponding harmonic analysis can be found in [47]. Reorganising each of these results into the sectors delineated by (5.27) yields the following sets of equations

$$\delta R_{\mu\nu} = \frac{m}{3}\mathring{g}_{\mu\nu}\epsilon^{\alpha\beta\sigma\rho}f_{\alpha\beta\sigma\rho} + 12m^2\mathring{g}_{\mu\nu}h^\alpha{}_\alpha - 12m^2h_{\mu\nu} \tag{5.47}$$

$$\delta R_{\mu n} = 6m^2h_{\mu n} + m\epsilon_\mu{}^{\alpha\beta\sigma}f_{n\alpha\beta\sigma} \tag{5.48}$$

$$\delta R_{mn} = 6m^2(h_{mn} - \mathring{g}_{mn}h^\alpha{}_\alpha) - \frac{m}{6}\mathring{g}_{mn}\epsilon^{\alpha\beta\sigma\rho}f_{\alpha\beta\sigma\rho} \tag{5.49}$$

$$\epsilon_{\beta\nu\rho\sigma}(\nabla_\mu f^{\mu\nu\rho\sigma} + \nabla_m f^{m\nu\rho\sigma}) + 9m\nabla_\beta(h^\alpha{}_\alpha - h^m{}_m) + 18m\nabla_m h^m{}_\beta = 0 \tag{5.50}$$

$$\nabla\mu f^{\mu n\rho\sigma} + \nabla_m f^{mn\rho\sigma} - 3m\epsilon^{\mu\alpha\rho\sigma}\nabla_\mu h^\alpha{}_\alpha = 0 \tag{5.51}$$

$$\nabla_\mu f^{\mu np\sigma} + \nabla_m f^{mnp\sigma=0} \tag{5.52}$$

$$\nabla_\mu f^{\mu npq} + \nabla_m f^{mnpq} = \frac{m}{4}\epsilon^{npqrstu}f_{rstu} \tag{5.53}$$

$$\nabla_m f_{\mu\nu\rho\sigma} + 4\nabla_{[\mu}f_{\nu\rho\sigma]m} = 0 \tag{5.54}$$

$$3\nabla_{[\mu}f_{\nu\rho]mn} + 2\nabla_{[m}f_{n]\mu\nu\rho} = 0 \tag{5.55}$$

$$2\nabla_{[\mu}f_{\nu]qrm} + 3\nabla_{[q}f_{rm]\mu\nu} = 0 \tag{5.56}$$

$$\nabla_\mu f_{pqrm} + 4\nabla_{[p}f_{qrm]\mu} = 0 \tag{5.57}$$

$$\nabla_{[m}f_{npqr]} = 0. \tag{5.58}$$

These are now the generic form for expansions around our FR vacuum of choice. To apply the KK ansatz we must substitute in the decomposition of our fields (5.2)

$$\hat{\Phi}_{MNK\ldots}(x,y) = \langle\hat{\Phi}_{MNK\ldots}(x,y)\rangle + \sum_{n=-\infty}^{\infty}\Big(\Phi(x)Y_\Phi(y)\Big)_{MNK\ldots}.$$

We once again demand independence on y at this energy and so we look to truncate the expansion to only the N zero modes, denoted $A \in \{1, ..., N\}$, of the harmonic modes $Y_\Phi(y)$

$$\hat{\Phi}_{MNK...}(x, y) = (\Phi^A(x, y)Y_A(x, y))_{MNK...}. \tag{5.59}$$

We also note that, interestingly, the zero modes form representations of SO(8) [32, 47]. We now apply this to fluctations about the metric. We first make the gauge transformation, which ultimately removes mixing between fields, and diagonalises them [48] (one is familiar with unitary used in the formulation of the standard model)

$$h_{MN} = h'_{MN} - \frac{1}{2}\mathring{g}_{MN}h^m{}_m. \tag{5.60}$$

The KK ansatz for the metric is then,

$$\begin{aligned} h'_{\mu\nu}(x, y) &= h_{\mu\nu}(x) \\ h'_{\mu n}(x, y) &= B^{[ij]}_\mu(x)K^{[ij]}_n(y) \\ h'_{mn}(x, y) &= S^{(ij)}(x)\Big[K^{[il]}_{(m}K^{[lj]}_{n)} - \frac{1}{9}\mathring{g}_{mn}K^{[il]}_pK^{[lj]p}\Big], \end{aligned} \tag{5.61}$$

where i, j, and l are indices running over S^7, from $1, ..., 8$. Here, $h_{\mu\nu}$ is a massless spin-2 particle, i.e. the graviton and is obviously a singlet with respect to the SO(8) indices; $B^{[ij]}_\mu$ is suggestively an element of the adjoint representation of SO(8); $S^{(ij)}$, in comparison to the pure AdS indices of $h_{\mu\nu}$, is a spacetime-scalar but an element of the traceless, symmetric representation of SO(8); and $K^{[ij]_m}$ are the 28 Killing vectors over S^7, as promised in the original ansatz (5.15). Note in the definition of h'_{mn}, that the scalars $S^{(ij)}$ are paired up with a symmetric, traceless combination of the Killing vectors, and that $B[ij]_\mu$ is paired to them in a 1-to-1 manner; the familiar face of gauge theory rears its head.

We once again make a gauge transformation to remove some of the rubble from the particle content after making the initial KK truncation

$$f_{\mu\nu\rho\sigma} = f'_{\mu\nu\rho\sigma} + \frac{3m}{2}\epsilon_{\mu\nu\rho\sigma}(h'^\alpha{}_\alpha - h^m{}_m) \tag{5.62}$$

$$f_{\mu\nu\rho q} = f'_{\mu\nu\rho q} + \frac{3m}{16}\epsilon_{\mu\nu\rho\sigma}\nabla^\sigma\nabla_q h^m{}_m \tag{5.63}$$

$$f_{\mu\nu pq} = f'_{\mu\nu pq} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho B^{\sigma[ij]}(x)K^{[il]}_{[p}K^{[lj]}_{q]}, \tag{5.64}$$

with $h'^\alpha{}_\alpha$ defined from (5.61). We then play the same game of pairing representations of SO(8) to complementary combinations of the Killing vectors and take the components of f_{MNPQ} to be

$$\begin{aligned} f'_{\mu\nu\rho\sigma}(x, y) &= 0 \\ f'_{\mu\nu\rho q}(x, y) &= 0 \\ f'_{\mu\nu pq}(x, y) &= 0 \\ f_{\mu npq}(x, y) &= \frac{1}{4}\partial_\mu P^{[ijkl]}(x)\big(\nabla_{[n}K^{[ij}\big)K_{q]}{}^{kl]} \\ f_{mnpq}(x, y) &= P^{[ijkl]}(x)\big(\nabla_{[m}K^{[ij}_n\big)\big(\nabla_pK^{kl]}{}_{q]}\big). \end{aligned} \tag{5.65}$$

The representation, $P^{[ijkl]}$, is comprised of psuedoscalars in the self-dual 4-vector **35** component of SO(8). This completes the ansatz, and so substituting the results into the low energy field equations (5.47)-(5.58) gives the results

$$\Delta_x h'_{\mu\nu} + 2\nabla_{(\mu}\nabla^\rho h'_{\nu)\rho} - \nabla_\mu\nabla_\nu h'^\rho{}_\rho + 24m^2h'_{\mu\nu} = 0 \tag{5.66}$$

$$\Delta_x B^{ij}_\mu + \nabla_\mu\nabla^\rho B^{ij}_\rho = 0 \tag{5.67}$$

$$(\Delta_x - 8m^2)S^{(ij)} = (\Delta_x - 8m^2)P^{[ijkl]} = 0, \tag{5.68}$$

where now Δ_x is the $4D$ Lichnerowicz operator. Equations (5.66) and (5.67) are the equations for infinitesimal fluctuations of the graviton and Yang-Mills-like field, indeed they correspond to infinitesimal perturbations of the corresponding field equations, to linear order [37]. The more unfamiliar field equation, (5.68), is a wave equation that is invariant under conformal transformations, which is what would be required to describe wave mechanics on a conformal background, such as the lightcone.

We now consider the gravitino field. The higher dimensional field equation is

$$\Gamma^{MNP}\tilde{D}_N\Psi_P = 0. \tag{5.69}$$

As before, we make a gauge transformation,

$$\Psi'_\mu = \Psi_\mu + \frac{1}{2}\gamma_5\gamma_\mu\Gamma^m\Psi_m, \tag{5.70}$$

and, under the FR ansatz, we have the following split in the field equations, just as for the bosonic sector

$$\gamma^{\mu\nu\rho}D_\nu\Psi'_\rho - \gamma^{\mu\rho}\gamma_5\Gamma^n D_n\Psi'_\rho - \frac{3m}{2}\gamma^{\mu\rho}\gamma_5\Psi'_\rho + \gamma^\mu\big(\mathring{g}^{np} - \frac{1}{2}\Gamma^n\Gamma^p\big)D_n\Psi_p - \frac{9m}{4}\gamma^\mu\Gamma^p\Psi_p = 0 \tag{5.71}$$

$$\begin{aligned}&\gamma_5\gamma^{\nu\rho}\Gamma^m D_\nu\Psi'_\rho + \gamma^\rho\Gamma^{mn}D_n\Psi'_\rho - \gamma^\nu\big(\mathring{g}^{mp} - \frac{1}{2}\Gamma^m\Gamma^p\big)D_\nu\Psi_p \\ &\qquad + \gamma_5\big(-\Gamma^{mnp} - 2\mathring{g}^{mp}\Gamma^n + 2\mathring{g}^{np}\Gamma^m\big)D_n\Psi_p + \frac{3m}{2}\gamma_5\Gamma^{mp}\Psi_p = 0.\end{aligned} \tag{5.72}$$

The KK ansatz for the fermions is then, to linear order in the field disturbances

$$\begin{aligned}\Psi'_\mu(x,y) &= \psi^I(x)\eta^I(y) \\ \Psi_m(x,y) &= \chi^{[IJK]}(x)\big[\eta_m^{[IJK]} + \frac{1}{9}\Gamma^m\Gamma^n\eta_n^{[IJK]}\big],\end{aligned} \tag{5.73}$$

where we have introduced the *Killing vector-spinor* $\eta_m^{IJK} \equiv \eta^{[I}\bar{\eta}^J\Gamma_m\eta^{K]}$ which satisfies the amalgamation of Killing spinor and vector properties[7] $\bar{D}_{(m}\eta_{n)} = 0$. This ansatz is completely analogous to the bosonic ansatz, but where we now consider Killing spinors as parameterising the extra dimensional components of the fields. Finally, inserting the spinorial ansatz into the associated field equations (5.71) and (5.72) delivers the following equations of motion

$$\gamma^{\mu\nu\rho}\bar{D}_\nu{\psi_\rho}^I = 0 \qquad \gamma^\mu D_\mu\chi^{IJK} = 0. \tag{5.74}$$

These are just the equations of motion for 8 spin-$\frac{3}{2}$ fields, and 56 spin-$\frac{1}{2}$ fields with covariant derivatives relating to motion in AdS_4. To conclude, the particle spectrum of the compactified and truncated theory on AdS_4 is shown in table 5.1. This is exactly the field content of the massless supergravity multiplet of $D = 4$, $\mathcal{N} = 8$. We have therefore yielded the same result described in section 4.1, but provided a physical mechanism for it as well. We now consider that each particle field sits inside a representation of SO(8). It can be shown that in fact the transformation laws of the fields are closed under the action of SO(8) [48, 32] and therefore the low energy theory has SO(8) gauge symmetry. In fact it was shown prior to this by de Wit and Nicolai [49] that $D = 4$, $\mathcal{N} = 8$ supergravity can be given local SO(8) symmetry. It is a natural conjecture to make that the low energy effective theory of $D = 11$ supergravity is exactly the supergravity of de Wit and Nicolai. Indeed, to linear order in perturbations the two sets of transformation laws for the particle spectra align. Returning briefly to the first example of Kaluza-Klein theory, $D = 5$ Einstein-Maxwell theory, we can see analogous phenomena occurring: a higher dimensional theory of spacetime symmetry is spontaneously compacted to the field content of a known lower dimensional theory. Finally, we note that miraculously, the ansatz is consistent. Both in terms of the lower dimensional field equations being closed as well as being solutions of the higher dimensional field equations [48, 50].

Spin	0	0	1/2	1	3/2	2
Field	S	P	χ	B_μ	ψ_μ	$g_{\mu\nu}$
Count	35	35	56	28	8	1

Table 5.1: Particle spectrum of compactified $D = 11$ supergravity

[7] Recall the a Killing vector satisfies $\nabla_{(i}K_{j)} = 0$.

Chapter 6

Conclusions

To summarise the results of this dissertation, the essence of Kaluza-Klein theory is gauge theory which warranted a thorough introduction of the most relevant aspects - as more abstract concepts are introduced, literacy on this matter is key. This chapter, 2, focused on notions of fibre bundles, (P, π, M, F), with an emphasis on specifically principal bundles and their associated vector bundles, as manifolds which are locally a product but, in general, globally different. This introduced the notion of a redundancy at each point on the base space M in the form of the fibre, F. We then defined covariant differentiation by way of a horizontal lift of curves - curves which are tangentially isomorphic to the base space curve, but with arbitrary choice of embedding. This ultimately led to the concept of Klein and Cartan geometry, which binds local group structure to spacetimes. This was followed up by the superspace analogue, where the core algebraic properties of the bosonic manifolds were transferred to construct a supersmooth supermanifold, locally isomorphic to superspace $(B_0^m \times B_1^n)$ with $B = B_0 \oplus B_1$ a Grassmann algebra. The presence of Grassmann-odd coordinates facilitates the action of a super-Lie group in a way that could be smoothly pulled back to a bosonic submanifold.

We next applied the tools of gauge theory to introduce and interpret consistent truncations. The fundamental idea was to produce a recognisable gauge theory in the low energy limit of a Lagrangian in higher-dimensional spacetime. This revolved around a few key postulates which comprised the 'Kaluza-Klein ansatz', which involved separating the lower dimensional space from any extra coordinates. Crucially, one of the manifolds needed to be compact as to be imperceptible to experiment. Applying this in general led to the Freund-Rubin ansatz. The principles of KK theory and gauge theory showed how $D = 11$ supergravity can spontaneously compactify to the product space $\mathrm{AdS}_4 \times S^7$ the symmetries of which yielded a gauged SO(8), $\mathcal{N} = 8$ supersymmetric, $D = 4$ theory of gravity.

We thus conclude our discussion of Kaluza-Klein methods on supergravity theories. The discussion has by no means been exhaustive and indeed focuses on the pre-superstring era of supergravity, instead we have placed a higher focus on the geometrical aspects of supergravity in the spirit of ordinary theory of gravitation. In one form or another, gauge theory has been at the heart of this discussion. To work with Kaluza-Klein mechanisms is to work with gauge theory, this is ultimately the beauty of the concept. Restructuring known gauge symmetries as higher dimensional spacetime symmetries is the essence of Kaluza-Klein truncations, be they with supersymmetries or not. An important point to stress is that Kaluza-Klein theories stipulate the physical existence of the extra dimensions, with associated dynamics that are out of reach of the energy scales of the everyday. Other methods of strict dimensional reduction outright discard any dependence on the extra dimensions, leaving them as a purely mathematical construction to facilitate physics. The same can be said about the existence of superspace, indeed Grassmann numbers cannot be measured which was related to the rheonomy principle. It is nevertheless clear that they do afford an elegant handling of supersymmetry and a generalisation of bosonic gauge theory. The simplicity in just by extracting the algebraic properties of differential geometry and applying them to even and odd-valued numbers yields a rigorous description of the dynamics of supersymmetry is very attractive. The prospect of further generalisations such as higher-Cartan geometry may reveal yet more interesting phenomena, without presupposing a 'target' such as supersymmetry, for instance.

Appendix A

A.1 Associated bundles as vector bundles

Proof. Take $p_0 \in P$ and $(p_0, v) \in P_V, \forall v \in V$. Then each v defines an equivalence class as the group action of G on P is free, by the definition of a principal bundle, i.e, if $v' = gv$ then $(p_0, v') \notin [p_0, v]$ as $(p_0, v') \simeq (p_0 g, v)$ and $p_0 g \neq p_0$ unless $g = e$, as the action of g is free. Define the set of all such equivalence classes $\{[p_0, v] \mid \forall v \in V\} \equiv B$. Define the projection map $\tau : B \to V, [p_0, v] \mapsto v$. Then τ is an isomorphism. Moreover, take $p_0 \in \pi^{-1}(x)$. Then $\pi(p_0 g) = x$ for all $g \in G$ and $\{(p_0 g, v) \mid \forall g \in G, v \in V\} \subseteq B$ as $(p_0 g, v) \simeq (p_0, gv) \in B$. Then $\pi_V([p_0, v]) = \pi(p_0 g) = x, \forall g \in G$ and so $\pi_V^{-1}(x) = B \simeq V$. □

A.2 Derivation of $D = 11$ vacuum field equations

We start by varying the only remaining parts of the action that contain the gauge potential - the Kinetic part and the Chern-Simons part

$$\begin{aligned}
\delta A_K &= -\frac{1}{12} e F^{MNPQ} \delta F_{MNPQ}, \quad \delta F_{MNPQ} = \partial_{[M} \delta A_{NPQ]} \\
&= \frac{1}{12} \delta A_{[NPQ} \partial_{M]} (e F^{MNPQ}) = \delta A_{NPQ} \partial_M (\mathbf{det}(\mathrm{e}) F^{MNPQ}) \\
&= \delta A_{NPQ} \partial_M (\sqrt{-g} F^{MNPQ}) = \delta A_{NPQ} \big[\partial_M (\sqrt{-g}) F^{MNPQ} + \sqrt{-g} \partial_M F^{MNPQ}\big] \\
&= \delta A_{NPQ} \big[\frac{1}{2} \sqrt{-g} g^{AB} (\partial_M g_{AB}) F_{MNPQ} + \sqrt{-g} \partial_M F^{MNPQ}\big] \\
&= \delta A_{NPQ} \sqrt{-g} \big[\partial_M F^{MNPQ} + \Gamma^A_{MA} F^{MNPQ}\big] = \delta A_{NPQ} \sqrt{-g} \nabla_M F^{MNPQ},
\end{aligned} \tag{A.1}$$

where integration by parts has been implied in between the first and second lines and we have used $\delta^\rho_\mu \Gamma^\mu_{\nu\rho} = \frac{1}{2} \delta^\rho_\mu g^{\mu\sigma} (\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}) = \frac{1}{2} g^{\rho\sigma} \partial_\nu g_{\rho\sigma}$ in going from the 4^{th} to the 5^{th} line. For the Chern-Simons term, it is easiest to temporarily keep it in differential form notation.

$$\begin{aligned}
\delta A_{CS} &= -\frac{\sqrt{2}}{3} e\big(d\delta A \wedge F \wedge A + F \wedge d\delta A \wedge A + F \wedge F \wedge \delta A\big) \\
&= \frac{\sqrt{2}}{3} e\big(d\delta A \wedge F \wedge A + d\delta A \wedge F \wedge A + F \wedge F \wedge \delta A\big) \\
&= \frac{\sqrt{2}}{3} e\big(2(-1)^{|F|+|A|-1} \delta A \wedge F \wedge dA + F \wedge F \wedge \delta A\big) \\
&= \frac{\sqrt{2}}{3} e\big(2\delta A \wedge F \wedge F + F \wedge F \wedge \delta A\big) \\
&= \frac{\sqrt{2}}{3} e\big((-1)^2 2F \wedge F \wedge \delta A + F \wedge F \wedge \delta A\big) \\
&= \sqrt{2} e (F \wedge F \wedge \delta A)
\end{aligned} \tag{A.2}$$

Putting these two parts together, leads to (up to normalisation) the field equation (5.26). (5.25) can be derived using similar methods, alongside the regular derivation for Einstein's field equations. □

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